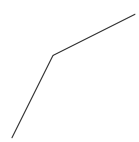
The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

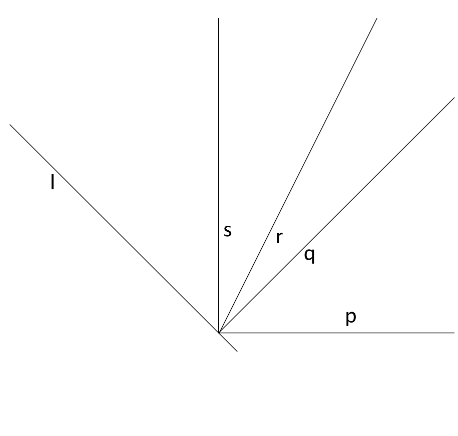
Please let me know if you spot any typos and I'll update things as soon as possible.

2.5.3: The odd coordinates in this problem were chosen to make it difficult to decide where the interior of the angle is. It's very close to a straight angle! If you check the slope of the line from $(17, -5)$ to $(123, 328)$ it rounds to 3.14151. The slope of the line from $(-96, -360)$ to $(17, -5)$ rounds to 3.14159. So if you draw a sketch, the bottom ray has a steeper slope. Overall that means the angle looks like this (not to scale!):



Hence the interior of the angle is to the lower right hand side of the angle. Because $(38, 61)$ is *above* the upper ray (check this!), the point is in the exterior.

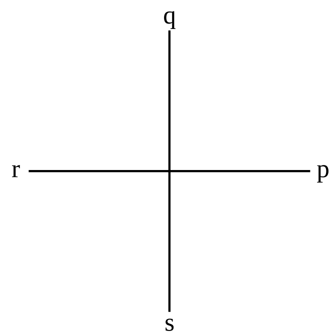
2.5.12: The situation looks something like this:



We can assume Proposition 10, but have to be careful about only applying it to two angles at once, each of which must meet along a common ray in the interior of the outer rays. Then

$$\begin{aligned}|\angle(p,s)| &= |\angle(p,q)| + (|\angle(q,s)|)\\ &= |\angle(p,q)| + (|\angle(q,r)| + |\angle(r,s)|)\end{aligned}$$

2.5.13: There are many possible answers. Here's one. Let p be the positive x-axis, q the positive y-axis, r the negative x-axis and s the negative y-axis:



With our definitions, $|\angle(p,s)| = \pi/2$, but:

$$|\angle(p,q)| + |\angle(q,r)| + |\angle(r,s)| = \pi/2 + \pi 2 + \pi 2 = 3\pi/2.$$

2.5.16: This problem was tricky, and the whole concept deserves a bit of an explanation. Suppose you have an angle with unit direction vectors U and V. Theorem 9 gives a nice formula to find the measure of the angle, based on the dot product $\langle U, V\rangle$. Alternatively, using Definition 14 we can label the measure as $\arccos\langle U, V\rangle$.

The problem is that the integral in the formula is hard to evaluate. It *does* have a value for any $\langle U, V\rangle$, but it's not at all clear how to find it. So we resort to sneakiness:

(1) We *define* π to be the measure of an angle for which $\langle U, V\rangle = -1$. That's Definition 11. As it happens, any such U and V represent a straight angle, as we would expect.
(2) In class, we asked what would happen if $\langle U, V\rangle = 0$. By comparing the resulting integral to the definition of π, we decided such an angle would have measure $\pi/2$. So that's why a right angle has measure $\pi/2$; see the discussion on the top of page 46.
(3) In Example 4, the authors pick U, V and W which happen to construct *two* angles which must combine to form one right angle. (Draw a picture of these vectors!) We know those angles have equal measure, since $\langle U, V\rangle = \langle V, W\rangle$, and the measure of an angle is entirely determined by the dot product of its (unit) direction indicators. So if two equal angles add to $\pi/2$, they must each measure $\pi/4$.

Following this pattern, you might try to pick U, V and W which form two angles which add up to $\pi/2$ such that $|\angle UOV|$ is twice as large as $|\angle VOW|$. Then one would have to be $\pi/3$ and the other $\pi/6$. But it turns out to be hard to show that one of those angles is twice the other; you can show that $\langle U, V\rangle$ and $\langle V, W\rangle$ are different, but there's no way to infer relative sizes of the angles from the values of those dot products.

I think it's easier to pick four points which define three angles:[1]

$$\begin{aligned} U &= (1, 0) \\ V &= (\sqrt{3}/2, 1/2) \\ W &= (1/2, \sqrt{3}/2) \\ X &= (0, 1) \end{aligned}$$

You can verify that $\langle U, V\rangle$, $\langle V, W\rangle$ and $\langle W, X\rangle$ are all equal, so the angles have equal measure. All three of them must add up to $\pi/2$ by problem 2.5.12. Hence each is one third of $\pi/2$ – also known as $\pi/6$. Then you can combine angles. $|\angle UOW|$ must be $2\pi/6 = \pi/3$, and $\langle U, W\rangle = 1/2$, which completes the first part of the problem.

For the second part, I'd probably define $Y = (-1/2, \sqrt{3}/2)$, show that $\langle X, Y\rangle = \sqrt{3}/2 = \langle U, V\rangle$, which imples that $|\angle XOY| = |\angle UOV| = \pi/6$, and then use Proposition 10 to show that $|\angle UOY| = 2\pi/3$. Since $\langle U, Y\rangle = -1/2$, I'd be done!

Clearly there are a few details here for you to fill in, but if you didn't figure out this problem on the homework, read through these solutions and sort things out before the exam.

Extra Problem: (a) The point here is that $1/\sqrt{1-t^2} > 0$ on $(-1, 1)$, so as t increases from -1 to 1 the area computed by the integral gets smaller. In other words, $\arccos(z)$ is a decreasing function.

Alternatively, using the Fundamental Theorem of Calculus:

$$\begin{aligned} \frac{d}{dz}\arccos(z) &= \frac{d}{dt}\int_z^1 \frac{1}{\sqrt{1-t^2}}\,dt \\ &= \frac{d}{dt} - \int_1^z \frac{1}{\sqrt{1-t^2}}\,dt \\ &= -\frac{1}{\sqrt{1-z^2}} < 0 \end{aligned}$$

So again, arccos is a decreasing function. Because it's decreasing on $[-1, 1]$ it must be one to one and therefore have an inverse. Its inverse function cos would flip-flop its inputs and outputs, so

$$\cos : [0, \pi] \to [-1, 1]$$

[1]How do I know which points to pick? This is a classic case of "knowing the answer before proving it." I'm using a bit of trignometry to find those points on the unit circle corresponding to the angles 0, $\pi/6$, $\pi/3$, and $\pi/2$.

(b) In the two-dimensional case, which suffices for our current needs,

$$\begin{aligned}\langle aU, bV\rangle &= \langle a(u_1, u_2), b(v_1, v_2)\rangle = \langle (au_1, au_2), (bv_1, bv_2)\rangle \\ &= au_1bv_1 + au_2bv_2 = (ab)u_1v_1 + (ab)u_2v_2 \\ &= ab\langle U, V\rangle\end{aligned}$$

(c) We have $U = A/||A||$ and $V = B/||B||$, so by part (b):

$$\langle U, V\rangle = \langle A/||A||, B/||B||\rangle = \frac{1}{||A||}\frac{1}{||B||}\langle U, V\rangle$$

(d) Let U and V be unit vectors in the same direction as A and B. Using part(c),

$$\theta = \arccos\left(\langle U, V\rangle\right) = \arccos\left(\frac{\langle A, B\rangle}{||A|| \cdot ||B||}\right)$$

Now, using the fact that arccos has an inverse function $\cos$, take $\cos$ of both sides:

$$\cos\theta = \frac{\langle A, B\rangle}{||A|| \cdot ||B||}$$

And rearrange to get the final formula.