The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

3.7.3: The barycentric coordinates give us the coefficients for a linear combination of the three points $A = (3, -2)$, $B = (4, -2)$, and $C = (4, -6)$:

$$(1, 1, -2)^{\triangle} = 1 \cdot (3, -2) + 2 \cdot (4, -2) - 2 \cdot (4, -6) = (3 + 8 - 8, -2 - 4 + 12) = (3, 6)$$

3.7.4: The conversion in this direction is more difficult. In class I gave two methods. The first is to set up a system of three equations and three unknowns. If $(r, s, t)^{\triangle ABC} = (0, 0)$, then:

$$\begin{aligned} 3r + 4s + 4t &= 0 \\ -2r - 2s - 6t &= 0 \\ r + s + t &= 1 \end{aligned}$$

The solution to this system gives $(0, 0) = (r, s, t)^{\triangle ABC} = (4, -5/2, -1/2)^{\triangle ABC}$

The other option is to use the more genera formula I developed using the matrix M whose colums are $B - A$ and $C - A$:

$$M = \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}$$

For a given point $X = (r, s, t)^{\triangle ABC}$ we know $\begin{bmatrix} s \\ t \end{bmatrix} = M^{-1}(X - A)$:

$$\begin{bmatrix} s \\ t \end{bmatrix} = M^{-1}\left(\begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1/4 \\ 0 & -1/4 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 11/4 \\ -7/4 \end{bmatrix}$$

Having found s and t, we calculate $r = 1 - s - t = 0$, so $(4, 5) = (0, 11/4, -7/4)^{\triangle ABC}$. The advantage to this second method is that you can find M^{-1} and then quickly use it for *both* conversions, instead of having to set up two separate systems of equations. But either method is fine if you get the correct answer.

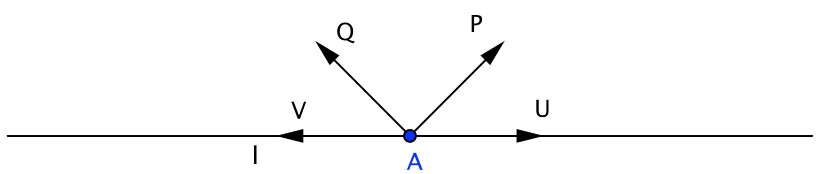
3.7.5: The two conditions in Proposition 15 follow very quickly from Definition 14; in the first, let $t = 0$, and in the second let $s = t = 1$. So the two conditions are special cases of the more general condition in the definition. We need to show that, given those two conditions, we can also show the more general statement is true. With a bit of careful thought and/or relabeling, that's not so bad. Let $s, t \in \mathbb{R}$ and $V, W \in \mathbb{R}^2$:

$$\begin{aligned}
\mathcal{T}(sV + tW) &= \mathcal{T}(A + B) && (\text{ for } A = sV,\ B = tW) \\
&= \mathcal{T}(A) + \mathcal{T}(B) && (\text{ by part (ii)}) \\
&= \mathcal{T}(sV) + \mathcal{T}(tW) && \\
&= s\mathcal{T}(V) + t\mathcal{T}(W) && (\text{ by part (i)})
\end{aligned}$$

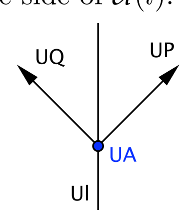
3.7.9: To show that an isometry $\mathcal{U}$ preserves half planes, it suffices to take any line l and two points P, Q on the same side of l – i.e. any two points in the half plane on that side of l – and show that $\mathcal{U}(P)$, $\mathcal{U}(Q)$ are on the same side of $\mathcal{U}(l)$. The whole proof rests on Theorem 6, specifically that isometries preserve lines, so $\mathcal{U}(l)$ is a line, and isometries preserve betweenness.

Suppose $\mathcal{U}(P)$ and $\mathcal{U}(Q)$ are on opposite sides of the line $\mathcal{U}(l)$. Then by definition, there exists a point on $\mathcal{U}(l)$ which is on the line segment between $\mathcal{U}(P)$ and $\mathcal{U}(Q)$. For consistency in notaion, I'll call that point $\mathcal{U}(R)$. [Because $\mathcal{U}$ is a bijection, I know there exists a unique point R which is sent to $\mathcal{U}(R)$.] Since $\mathcal{U}$ and $\mathcal{U}^{-1}$ preserve betweenness, R is between P and Q. But then P and Q were on opposite sides of the line l to begin with! Hence if P and Q are on the same side of l, $\mathcal{U}(P)$ and $\mathcal{U}(Q)$ must be on the same side of $\mathcal{U}(l)$.

There are other possible approaches to this problem. Many people used the fact that $\mathcal{U}$ must preserve angles and wrote a proof along these lines. Take a point A on the line l, and suppose P and Q are on opposite sides of l, as in the following picture.



You can then say that $|\angle PAQ| = |\angle \mathcal{U}(P)\mathcal{U}(A)\mathcal{U}(Q)|$. However, this isn't quite enough to prove $\mathcal{U}(P)$ and $\mathcal{U}(Q)$ are on the same side of $\mathcal{U}(l)$. You could have a situation like this:



To avoid this possibility, you need to keep track of *all* the angles in the original picture, and their sum:

$$|\angle UAP| + |\angle PAQ| + |\angle QAV| = \pi$$

An isometry $\mathcal{U}$ will send these "side-by-side" angles to three more angles which are side by side and still add up to π. You can use that fact as a basis for arguing that the half plane is preserved.

3.7.10: Any counter example will do. Here's a simple one: $\mathcal{U}(X) = X + (3, 2)$, $a = b = 1$, $P = (0, 0)$, $Q = (1, 0)$. Then:

$$\mathcal{U}(aP + bQ) = \mathcal{U}(\,(1, 0)\,) = (4, 2)$$

which is quite different than

$$a\mathcal{U}(P) + b\mathcal{U}(Q) = (3, 2) + (4, 2) = (7, 4)$$

3.7.11: The back of the book shows how to extend Lemma 4 from a combination of two vectors to three vectors. (If you have trouble following the proof, I've walked through it a number of times with students who were working on this assignment, and would be happy to do so again in office hours.) So to complete the proof by induction, we assume

$$\mathcal{U}(a_1P_1 + \cdots + a_{n-1}P_{n-1}) = a_1\mathcal{U}(P_1) + \cdots + a_{n-1}\mathcal{U}(P_{n-1})$$

whenever $a_1 + \cdots + a_{n-1} = 1$, and now have to show the same is true with a combination of n vectors instead:

$$\mathcal{U}(a_1P_1 + \cdots + a_{n-1}P_{n-1} + a_nP_n) = a_1\mathcal{U}(P_1) + \cdots + a_{n-1}\mathcal{U}(P_{n-1}) + a_n\mathcal{U}(P_n)$$

if $a_1 + \cdots + a_{n-1} + a_n = 1$. First I note that the a_i's can't all equal 1, or else if $a_1 + \cdots + a_{n-1} + a_n = n \neq 1$. So it's safe for me to assume $a_n \neq 1$; if not, I would just swap a_n and P_n with whatever term has an $a_i \neq 1$. Now I group terms together:

$$\begin{aligned}
\mathcal{U}\left(a_1P_1 + \cdot + a_{n-1}P_{n-1} + a_nP_n\right) &= \mathcal{U}\left(1 \cdot (a_1P_1 + \cdot + a_{n-1}P_{n-1}) + a_nP_n\right) \\
&= \mathcal{U}\left((1 - a_n) \cdot \frac{(a_1P_1 + \cdot + a_{n-1}P_{n-1})}{1 - a_n} + a_nP_n\right) \\
&= \mathcal{U}\left((1 - a_n) \cdot \left(\frac{a_1}{1 - a_n}P_1 + \cdot + \frac{a_{n-1}}{1 - a_n}P_{n-1}\right) + a_nP_n\right)
\end{aligned}$$

The reason I multiplied and divided by $(1 - a_n)$ is that now I can apply Lemma 4: I have two vectors (one of which is a combination of $n-1$ vectors) multiplied by coefficients which add to 1. Hence:

$$\mathcal{U}\left(a_1P_1 + \cdot + a_{n-1}P_{n-1} + a_nP_n\right) = (1 - a_n)\mathcal{U}\left(\frac{a_1}{1 - a_n}P_1 + \cdot + \frac{a_{n-1}}{1 - a_n}P_{n-1}\right) + a_n\mathcal{U}(P_n)$$

Now notice that

$$\frac{a_1}{1 - a_n} + \cdots + \frac{a_{n-1}}{1 - a_n} = \frac{a_1 + \cdots + a_{n-1}}{1 - a_n} = \frac{1 - a_n}{1 - a_n} = 1$$

So we can apply our assumed result about a linear combination of $n-1$ vectors and we'll be finished. To tie everything together I will write the entire string of calculations from beginning to end:

$$\begin{aligned}
\mathcal{U}\left(a_1P_1 + \cdot + a_{n-1}P_{n-1} + a_nP_n\right) &= \mathcal{U}\left(1 \cdot (a_1P_1 + \cdot + a_{n-1}P_{n-1}) + a_nP_n\right) \\
&= \mathcal{U}\left((1-a_n) \cdot \frac{(a_1P_1 + \cdot + a_{n-1}P_{n-1})}{1-a_n} + a_nP_n\right) \\
&= \mathcal{U}\left((1-a_n) \cdot \left(\frac{a_1}{1-a_n}P_1 + \cdot + \frac{a_{n-1}}{1-a_n}P_{n-1}\right) + a_nP_n\right) \\
&= (1-a_n)\mathcal{U}\left(\frac{a_1}{1-a_n}P_1 + \cdot + \frac{a_{n-1}}{1-a_n}P_{n-1}\right) + a_n\mathcal{U}(P_n) \\
&= (1-a_n)\left(\frac{a_1}{1-a_n}\mathcal{U}(P_1) + \cdot + \frac{a_{n-1}}{1-a_n}\mathcal{U}(P_{n-1})\right) + a_n\mathcal{U}(P_n) \\
&= a_1\mathcal{U}(P_1) + \cdots + a_{n-1}\mathcal{U}(P_{n-1}) + a_n\mathcal{U}(P_n)
\end{aligned}$$

Whew!