The following is a non-comprehensive list of solutions. I've tried to explain enough that you can figure out any mistakes you might have made, but I haven't written out every excruciating detail. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

Problem A: As described in class, there are three steps to finding the measure of any angle which is described by two vectors A and B which start at the vertex C:

(1) Find unit vectors $U = A/||A||$ and $B = B/||B||$ which point in the direction of A and B.

(2) Calculate the dot product $\langle U, V\rangle$.

(3) Then $|\angle ACB| = |\angle UCV| = \arccos(\langle U, V\rangle) = \int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}}$.

In this problem, $C = (0, 0) = O$ and U is already a unit vector, but X is not. So we compute:

$$W = \frac{X}{||X||} = \frac{(3,4)}{5} = (3/5, 4/5)$$

Since $\langle U, W\rangle = 3/5$, we can conclude that $|\angle UOX| = |\angle UOW| = \arccos(3/5)$, although that turns out to be irrelevant to solving the problem. You're asked to find a vector $V = (v_1, v_2)$ such that $|\angle UOV| = |\angle VOW|$. Assuming V is a unit vector, we have

$$|\angle UOV| = \arccos(\langle U, V\rangle)$$
$$|\angle VOW| = \arccos(\langle V, W\rangle)$$

So all we need to do is find a unit vector $V = (v_1, v_2)$ such that $\langle U, V\rangle = \langle V, W\rangle$. We calculate:

$$\begin{aligned}
\langle U, V\rangle &= \langle V, W\rangle \\
\langle (1,0), (v_1, v_2)\rangle &= \langle (v_1, v_2), (3/5, 4/5)\rangle \\
v_1 &= \frac{3}{5}v_1 + \frac{4}{5}v_2 \\
\frac{2}{5}v_1 &= \frac{4}{5}v_2 \\
v_1 &= 2v_2
\end{aligned}$$

In fact any vector V in the first quadrant[1] for which $v_1 = 2v_2$ works as a solution, such as $V = (2, 1)$. But if you want the unit vector it would be $V = (2, 1)/||(2, 1)|| = \left(2/\sqrt{5}, 1/\sqrt{5}\right)$.

1.6.22: We're asked to convert each line to its *special* parametric and normal form, which means our direction indicators and coefficient vectors need to be normalized.

(ii) $(3, 4)+t(-1, 2)$ represents a line through the point $(3, 4)$ in the direction of $(-1, 2)$. Normalizing the direction indicator gives the special parametric form:

$$(3,4) + t\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

The vector $(2, 1)$ is perpendicular to the direction indicator and hence perpendicular to the line. So the normal form of the line would be:

$$\{\langle (2,1), X - (3,4)\rangle = 0\}$$

[1] We don't want $V = (-2, -1)$, since that wouldn't be in the angle $\angle UOW$.

Normalizing the coefficient vector gives the special normal form:

$$\{\langle(2/\sqrt{5},1/\sqrt{5}),X-(3,4)\rangle=0\}$$
$$\{\langle(2/\sqrt{5},1/\sqrt{5}),X-(3,4)\rangle=0\}$$
$$\{\langle(2/\sqrt{5},1/\sqrt{5}),X\rangle-\langle(2/\sqrt{5},1/\sqrt{5}),(3,4)\rangle=0\}$$
$$\{\langle(2/\sqrt{5},1/\sqrt{5}),X\rangle=10/\sqrt{5}\}$$

(iii) For $(0,0)+t(-7,24)$, we have:

$$\text{Special Parametric:} \quad t\left(-\frac{7}{25},\frac{24}{25}\right)$$
$$\text{Special Normal:} \quad \{\langle(24/25,7/25),X\rangle=0$$

3.6.6: Our definition of a 2×2 orthogonal matrix is that the two columns are both unit vectors and are orthogonal, so you had to check both of those conditions. Matrices (i) and (ii) are orthogonal, and their inverses are their transposes, by Proposition 17 in Chapter 3.

Many people said "A matrix M is orthogonal if $MM^T=I$, which is true, but it's not our definition and it's not what Proposition 17 says.[2]

3.7.10: If $(r,s,t)^{\triangle ABC}=(0,0)$, then [using $A=(3,-2)$, $B=(4,-2)$ and $C=(4,-6)$]:

$$\begin{aligned} 3r+4s+4t &= 0\\ -2r-2s-6t &= 0\\ r+s+t &= 1 \end{aligned}$$

The solution to this system gives $(0,0)=(r,s,t)^{\triangle ABC}=(4,-5/2,-1/2)^{\triangle ABC}$

Similarly, $(4,5)=(0,11/4,-7/4)^{\triangle ABC}$.

3.7.12: Any counterexample will do. Here's a simple one: $\mathcal{U}(X)=X+(3,2)$, $a=b=1$, $P=(0,0)$, $Q=(1,0)$. Then:

$$\mathcal{U}(aP+bQ)=\mathcal{U}(\,(1,0)\,)=(4,2)$$

which is quite different than

$$a\mathcal{U}(P)+b\mathcal{U}(Q)=(3,2)+(4,2)=(7,4)$$

3.7.23: To me this problem is most difficult because of the notation, especially $\mathcal{U}$ and U. Also, Y is the *input* for the original isometry and X is the output. (Hence X is the input of $\mathcal{U}^{-1}$ and Y is the output of $\mathcal{U}^{-1}$.) We may assume U is a 2×2 orthogonal matrix in the given formula; to avoid overuse of the letter, I will replace it with M:

$$X=\mathcal{U}(Y)=MY+P$$

To find the inverse, solve that matrix/vector equation for Y in terms of X. Not every matrix M has an inverse, so it must be mentioned that M^{-1} exists because M is orthogonal. (That's Proposition 17 in Chapter 3.)

$$X=MY+P$$
$$M^{-1}X=M^{-1}MY+M^{-1}P=Y+M^{-1}P$$
$$M^{-1}X-M^{-1}P=Y$$

So $Y=\mathcal{U}^{-1}(X)=M^{-1}X-M^{-1}P$.

[2] Perhaps not coincidentally, it *is* the second sentence in Wikipedia's entry on orthogonal matrices...

3.7.25: We have two isometries,

$$\mathcal{U}(X) = MX + P, \text{ where } M = \frac{1}{53}\begin{bmatrix} 28 & 45 \\ -45 & 28 \end{bmatrix}, P = \begin{bmatrix} -8 \\ 25 \end{bmatrix}$$
$$\mathcal{V}(X) = NX + Q, \text{ where } N = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The compositions are:

$$\begin{aligned}
\mathcal{U} \circ \mathcal{V}(X) &= \mathcal{U}(\mathcal{V}(X)) \\
&= M(NX + Q) + P \\
&= (MN)X + (MQ + P) \\
&= \frac{1}{106}\begin{bmatrix} (45 + 28\sqrt{3}) & (28 - 45\sqrt{3}) \\ (28 - 45\sqrt{3}) & (-45 - 28\sqrt{3} \end{bmatrix} X + \begin{bmatrix} -379/53 \\ 1353/53 \end{bmatrix} \\
\mathcal{V} \circ \mathcal{U}(X) &= \mathcal{V}(\mathcal{U}(X)) \\
&= N(MX + P) + Q \\
&= (NM)X + (NP + Q) \\
&= \frac{1}{106}\begin{bmatrix} (-45 + 28\sqrt{3}) & (28 + 45\sqrt{3}) \\ (28 + 45\sqrt{3}) & (45 - 28\sqrt{3} \end{bmatrix} X + \begin{bmatrix} \left(\frac{25}{2} - 4\sqrt{3}\right) \\ \left(-3 - \frac{25\sqrt{3}}{2}\right) \end{bmatrix}
\end{aligned}$$