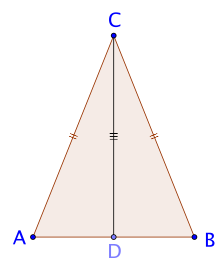
The following is a non-comprehensive list of solutions. I've tried to explain enough that you can figure out any mistakes you might have made, but I haven't written out every excruciating detail. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work. Please let me know if you spot any typos and I'll update things as soon as possible.

4.11.43: If $\triangle ABC$ is isosceles, then the incenter is collinear with both the centroid and circumcenter. To prove this, consider the isosceles triangle below.



Suppose $\overline{CD}$ is an angle bisector. Then $|\angle ACD| = |\angle BCD|$, and by SAS we have $\triangle ACD \cong \triangle BCD$. That lets us conclude:

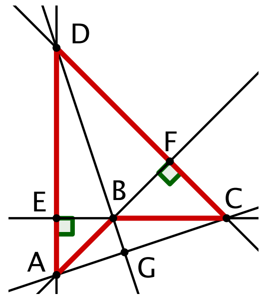
- $\overline{AD} \cong \overline{BD}$, so D is the midpoint of $\overline{AB}$ and CD is a median.
- $|\angle ADC| = |\angle BDC|$ and, since they add up to π, each is $\pi/2$. So $\overline{CD}$ is an altitude and–since D is the midpoint–a perpendicular bisector.

Since $\overline{CD}$ is a median, perpendicular bisector and angle bisector, the centroid, circumcenter and incenter must all lie on it; hence they are collinear. This line will in fact be the Euler line unless the triangle is equilateral (a special case of isosceles), in which case all of the points coincide and there is no Euler line.

For full credit you also needed to say a few words about what goes wrong with other triangles. As mentioned, in equilateral triangles the incenter, centroid and circumcenter are the same, and there is no Euler line. Proving that the incenter is not on the Euler line for scalene triangles is tricky (or just tedious, if you use barycentric coordinates), but any sort of explanation or example to show you had thought about the issue would suffice.

6.6.57: There are a few different approaches you could take to this problem. Notice that the problem refers to #56, which in turn refers to #55. So if you were really stuck, looking at those problems could have been useful. The proof I sketch here is based on some of the ideas in those problems.

First, consider an orthcentric quadrangle $ABCD$, like the dart I suggested in class and shown below. Notice that I've drawn the lines between every pair of vertices, not just the adjacent ones. The actual quadrilaterial $ABCD$ is highlighted in red.



We know $\overline{AB} \perp \overline{CD}$ and $\overline{BC} \perp \overline{AD}$, so I've labeled the points of intersection of these lines and drawn in the right angles where appropriate. However, there's one more right angle: I claim $\overline{AC} \perp \overline{BD}$. One way to prove this is to use Problem 6.11.16 from the previous homework assignment. In the quadrangle $AEBG$, $|\angle AEB| + |\angle AGB| = \pi$, and since $|\angle AEB| = \pi/2$, we conclude $|\angle AGB| = \pi/2$ as well.

To summarize, we have the following pairs of perpendicular lines:

$$\overline{AB} \perp \overline{CD}$$
$$\overline{BC} \perp \overline{AD}$$
$$\overline{AC} \perp \overline{BD}$$

Now we're ready to start our proof that, given an orthcentric quadrangle, any vertex is the orthocenter of the other three. Choose any three points–say A, B and C, although the following proof works similarly no matter which three we choose. The orthocenter of $\triangle ABC$ is the intersection of its altitudes:

- The altitude from A is the line through A which is perpendicular to $\overline{BC}$. From our list above, that's $\overline{AD}$.
- The altitude from B is the line through B which is perpendicular to $\overline{AC}$. From our list above, that's $\overline{BD}$.
- The altitude from C is the line through C which is perpendicular to $\overline{AB}$. From our list above, that's $\overline{CD}$.

Notice all three of the altitudes go through D, which is therefore the orthocenter of $\triangle ABC$.

In the other direction, it suffices to start with any arbitrary $\triangle ABC$, let D be its orthocenter, and show that $ABCD$ is an orthocentric quadrangle, i.e. that opposite sides are perpendicular:

$\overline{AB} \perp \overline{CD}$: This is true because $\overline{CD}$ is an altitude – it includes the vertex C and the orthocenter. So it must be perpendicular to the side of $\triangle ABC$ which is opposite C, namely $\overline{AB}$.

$\overline{BC} \perp \overline{AD}$: Similarly, this is true because $\overline{AD}$ is an altitude and must be perpendicular to $\overline{BC}$.

Hence $ABCD$ is an orthocentric quadrangle.

7.9.26: Let $\mathcal{C}_C(X) = -X + 2C$ be a central inversion about a center C, and $\mathcal{T}_P(X) = X + P$ be a translation by P. Then:

(i) $\mathcal{T}_P \circ \mathcal{C}_C(X) = \mathcal{T}_P(-X + 2C) = -X + 2C + P = -X + 2\left(C + \frac{1}{2}P\right)$ This is the formula for a central inversion centered at the point $\left(C + \frac{1}{2}P\right)$.

(ii) $\mathcal{C}_C \circ \mathcal{T}_P(X) = \mathcal{C}_C(X+P) = -X - P + 2C = -X + 2\left(C - \frac{1}{2}P\right)$ This is the formula for a central inversion centered at the point $\left(C - \frac{1}{2}P\right)$.

7.9.27: For two points C and D, we have

$$\mathcal{C}_D(X) \circ \mathcal{C}_C(X) = \mathcal{C}_D(-X + 2C) = X - 2C + 2D$$

Which is a translation by $P = -2C + 2D = 2(D - C)$, twice the vector from the center of the first central inversion to the center of the second. Note that if $C = D$, $P = 0$ and the composition will result in the identity as expected. (Any central inversion is its own inverse.)

7.9.30: (i) The two lines are perpendicular and intersect at $C = \left(\frac{47}{68}, \frac{1}{68}\right)$. By our work in class, that means the composition of reflections across the two lines is a central inversion, centered at C:

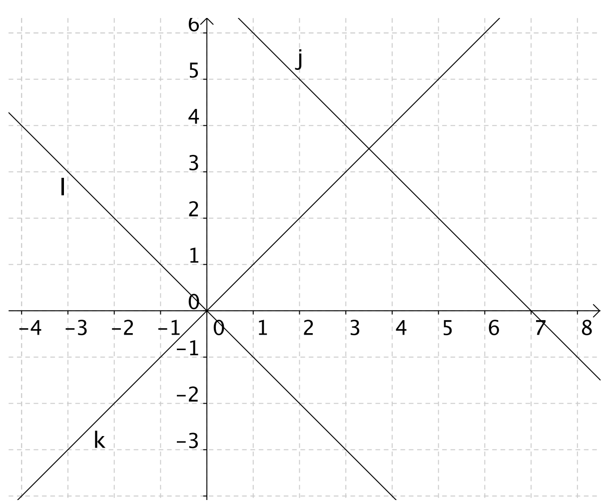
$$\mathcal{C}_{(47/68,1/68)}(X) = -X + 2(47/68, 1/68)$$

(ii) The two lines intersect at $(2, 0)$ when $t = 0$ and $s = -1/2$. The angle α from the first line to the second is about $59.03°$; from our work in class, the net result is a rotation about the point $(2, 0)$ by an angle of 2α. As with earlier homework problems, we can draw triangles on these lines to determine exact values of $\cos 2\alpha$ and $\sin 2\alpha$ in order to write down our rotation matrix. The resulting matrix formula is:

$$\mathcal{R}(X) = \frac{1}{17}\begin{bmatrix} -8 & -15 \\ 15 & -8 \end{bmatrix}\left(X - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{or} \quad \frac{1}{17}\begin{bmatrix} -8 & -15 \\ 15 & -8 \end{bmatrix} X + \begin{bmatrix} 50/17 \\ -30/17 \end{bmatrix}$$

(iv,v) These two problem involve three lines, which I've named as follows and graphed with GeoGebra:

$$j : (2, 5) + s(3, -3)$$
$$k : \langle(-3, 3), X\rangle = 0$$
$$l : \langle(-3, -3), X\rangle = 0$$



In part (iv) we are working with j and k, lines which meet at a right angle at $C = (7/2, 7/2)$. (Verify this!) Hence either $\mathcal{M}_j \circ \mathcal{M}_k$ or $\mathcal{M}_k \circ \mathcal{M}_j$ will be a central inversion about the point

C. (Technically you could argue that one will be a rotation by π and the other a rotation by $-\pi$, but the end result is the same.) The formula is given by

$$\mathcal{C}_C(X) = -X + 2C = -X + 2C = -X + 2\begin{bmatrix} 7/2 \\ 7/2 \end{bmatrix} = -X + \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

In part (v) we are working with j and l, lines which are parallel. The vector $U = (7/2, 7/2)$ is perpendicular to both and stretches exactly from $O \in l$ to $(7/2, 7/2) \in j$. (Check all of this!) The composition of two reflections across parallel lines is a translation in the direction from the first to the second, with a distance twice that between the lines. Thus:

$$\mathcal{M}_l \circ \mathcal{M}_j(X) = \mathcal{T}_{-2U}(X) = X - 2U = X - \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

7.9.34: Let ℓ be the line $y = -4$ and k be $y = 4$. Then the formulas for the translations and reflections used in this problem are as follows. [Check these! I've written them out with x's and y's instead of matrices and vectors. Ask me if you're not sure how to get from your matrix formulas to this form.]

$$\begin{aligned} \mathcal{T}_{(5,0)}(x, y) &= (x + 5, 0) \\ \mathcal{T}_{(-5,0)}(x, y) &= (x - 5, 0) \\ \mathcal{M}_\ell(x, y) &= (x, -8 - y) \\ \mathcal{M}_k(x, y) &= (x, 8 - y) \end{aligned}$$

We're asked to figure out the following composition:

$$\begin{aligned} \mathcal{T}_{(-5,0)} \circ \mathcal{M}_k \circ \mathcal{T}_{(5,0)} \circ \mathcal{M}_\ell(x, y) &= \mathcal{T}_{(-5,0)} \circ \mathcal{M}_k \circ \mathcal{T}_{(5,0)}(x, -8 - y) \\ &= \mathcal{T}_{(-5,0)} \circ \mathcal{M}_k(x + 5, -8 - y) \\ &= \mathcal{T}_{(-5,0)}(x + 5, 8 - (-8 - y)) \\ &= (x + 5 - 5, 8 - (-8 - y)) \\ &= (x, y + 16) \end{aligned}$$

So the overall net effect is a translation by $(0, 16)$, or in our other notation,

$$\mathcal{T}_{(-5,0)} \circ \mathcal{M}_k \circ \mathcal{T}_{(5,0)} \circ \mathcal{M}_\ell = \mathcal{T}_{(0,16)}$$

8.4.3: You can almost use the formula given in the book for a circle inversion, with one exception: that formula assumes the mirror is centered at the origin, whereas our circle is centered at $(-8, 13)$. So we need to first move everything so the center is at the origin, then invert, and then move it back:

$$\mathcal{I}(X) = \begin{cases} \frac{\rho^2}{||X-C||^2}(X - C) + C, & X \neq C, \infty \\ \infty & X = C \\ C & X = \infty \end{cases}$$

where $C = (-8, 13)$ and $\rho = 29$ in our case. By my quick calculations, this yields:

$$\mathcal{I}(0,0) = \left(\frac{4864}{233}, \frac{-7904}{233} \right)$$

$$\mathcal{I}(12,-8) = (12,-8) \text{ (this point is on the mirror, so it stays fixed!)}$$

$$\mathcal{I}(\infty) = (-8,13) \text{ (the point at } \infty \text{ always goes to the circle center)}$$

$$\mathcal{I}(8,-13) = \left(\frac{1500}{233}, \frac{-4875}{233} \right)$$