Math 5335 Fall 09 Homework #5 Solutions

These solutions were written using *Mathematica* so that I could make use of its computational and graphing abilities. Don't worry about understanding any of the computer commands interspersed throughout the solutions, but let me know if you have trouble understanding any of the answers or need help with the process.

8.3 #4

(i) A circle of radius 1/2 centered at O

(ii) A circle of radius 1/3 centered at O

(iii) The line $y = -1/2$

(iv) A circle centered at (0, -1/2) of radius 1/2.

(v) This line is fixed - its image is the same line.

9.9 #7

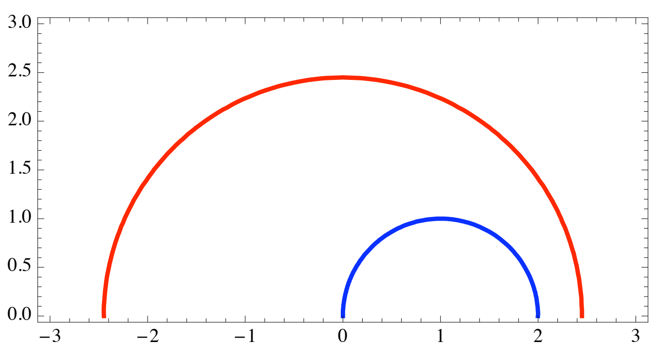
These lines do not intersect, as you can see in the picture below. To prove this, you need to solve the system of equations

$x^2 + y^2 = 6$
$(x-1)^2 + y^2 = 1$

You can show algebraically that there are no values of x and y which simultaneously satisfy both equations, proving the lines do not meet. (Ask me for help if you have trouble with this system.)

In[63]:=
```
ContourPlot[{x^2 + y^2 == 6, (x - 1)^2 + y^2 == 1}, {x, -3, 3}, {y, 0, 3},
 AspectRatio → Automatic, ContourStyle → {{Thick, Red}, {Thick, Blue}}]
```

Out[63]=



9.9 #8

These lines **do** intersect, as you can see in the picture below.

```
ContourPlot[{x^2 + y^2 == 6,  (x + 4)^2 + y^2 == 9}, {x, -7, 3}, {y, 0, 4},
 AspectRatio → Automatic, ContourStyle → {{Thick, Red}, {Thick, Blue}}]
```

To find the point of intersection, solve the system of equations:

$x^2 + y^2 = 6$
$(x+4)^2 + y^2 = 9$

Mathematica can do that:

```
Solve[x^2 + y^2 == 6 && (x + 4)^2 + y^2 == 9, {x, y}]
```

Notice that *Mathematica* gives us two solutions; that's because if we drew the entire circles, they would intersect again in the lower half plane. We're only interested in the intersection with a postive y value, which is

$$\left(-\frac{13}{8}, \frac{\sqrt{215}}{8}\right)$$

▪ 9.9 #26

Mathematica can handle our messy distance formula from section 10.1:

```
In[64]:= dist[a_, b_, c_, d_] =
          Log[ ((c - a)^2 + d^2 + b^2) / (2 d b) + Sqrt[(((c - a)^2 + d^2 + b^2) / (2 d b))^2 - 1]]
```

Out[64]= $\text{Log}\left[\frac{b^2 + (-a+c)^2 + d^2}{2\,b\,d} + \sqrt{-1 + \frac{\left(b^2 + (-a+c)^2 + d^2\right)^2}{4\,b^2\,d^2}}\right]$

Now let's plug our points into this distance function; I'll also ask *Mathematica* for a decimal approximation of each answer:

```
In[65]:= dist[-2, 4, 2, 4]
         N[%]
```

Out[65]= $\text{Log}\left[\frac{3}{2} + \frac{\sqrt{5}}{2}\right]$

Out[66]= 0.962424

```
In[67]:= dist[0, 4, 2, 4]
         N[%]
```

Out[67]= $\text{Log}\left[\frac{9}{8} + \frac{\sqrt{17}}{8}\right]$

Out[68]= 0.494933

```
In[69]:= dist[-2, 4, 0, 4]
         N[%]
```

Out[69]= $\text{Log}\left[\frac{9}{8} + \frac{\sqrt{17}}{8}\right]$

Out[70]= 0.494933

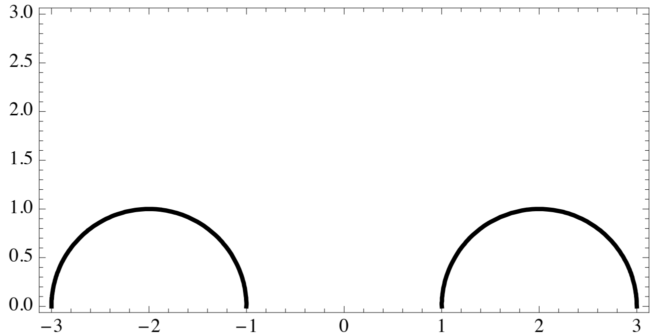
■ 10.5 #7

One of our lines has Poincare Direction Indicators (PDIs) $(-3, 0)$ and $(-1, 0)$. The other has PDIs $(1, 0)$ and $(3, 0)$. We want the four lines which start at one of the first two PDIs and end at one of the last two:

Here's a graph of the original two lines:

In[71]:=
```
ContourPlot[{(x - 2) ^2 + y^2 == 1, (x + 2) ^2 + y^2 == 1},
 {x, -3, 3}, {y, 0, 3}, AspectRatio → Automatic, ContourStyle → Thick]
```

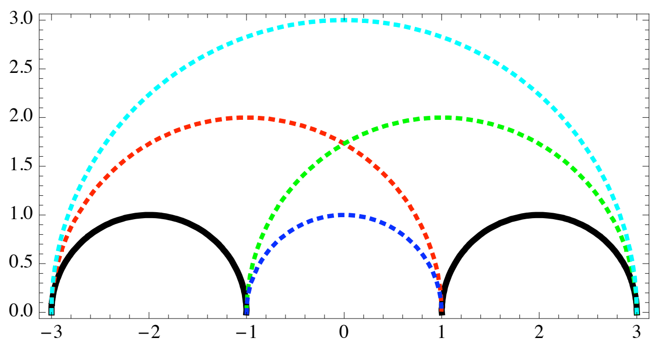
Out[71]=



Here's a graph with the other lines added on:

In[72]:=
```
ContourPlot[{(x - 2) ^2 + y^2 == 1,
   (x + 2) ^2 + y^2 == 1,
   (x + 1) ^2 + y^2 == 4,
   (x - 1) ^2 + y^2 == 4,
   x^2 + y^2 == 1,
   x^2 + y^2 == 9}, {x, -3, 3}, {y, 0, 3}, AspectRatio → Automatic,
 ContourStyle → {Thickness[0.01], Thickness[0.01], {Thick, Red, Dotted},
   {Thick, Green, Dotted}, {Thick, Blue, Dotted}, {Thick, Cyan, Dotted}}]
```

Out[72]=

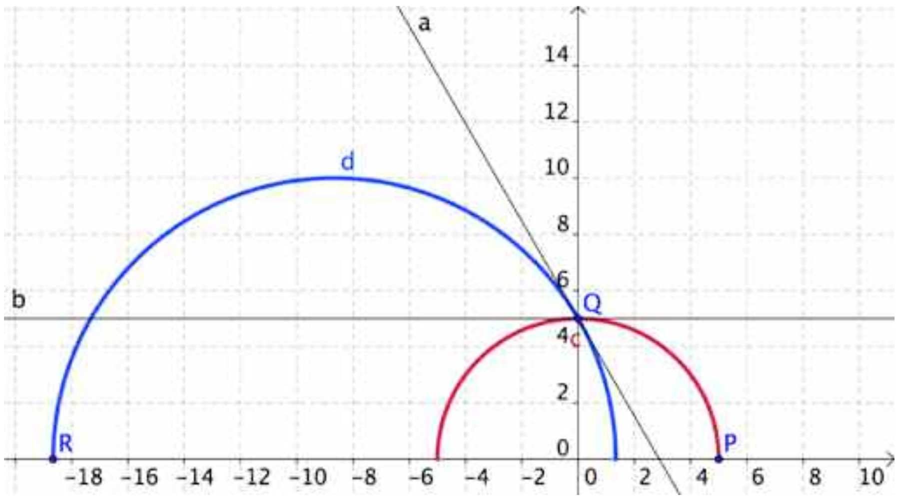


The equations are as follows:

$x^2 + y^2 = 1$
$x^2 + y^2 = 9$
$(x-1)^2 + y^2 = 4$
$(x+1)^2 + y^2 = 4$

■ 10.5 #8

Using the formulas on pp191-192 in Chapter 9, essentially restated as Proposition 4 in Chapter 10, we can find the equations of the lines through (0,5) directed by the given PDI's. Here are the graphs, imported from GeoGebra. I added the tangent lines to the circles at the vertex; the angle we're measuring is the angle where the Q is between the lines:



Mathematica can handle the long formula in Theorem 7:

```
In[73]:= angle[g_, h_, a_, u_] =
          ArcCos[((h^2 + (a - g) (u - g))^2 - h^2 (u - a)^2) / (((a - g)^2 + h^2) ((u - g)^2 + h^2))]
```

Out[73]= $\text{ArcCos}\left[\frac{-h^2 (-a+u)^2 + \left(h^2 + (a-g)(-g+u)\right)^2}{\left((a-g)^2 + h^2\right)\left(h^2 + (-g+u)^2\right)}\right]$

In our case, we get

```
In[74]:= angle[0, 5, 5, -10 - 5 Sqrt[3]]
```

Out[74]= $\text{ArcCos}\left[\frac{-25\left(-15 - 5\sqrt{3}\right)^2 + \left(25 + 5\left(-10 - 5\sqrt{3}\right)\right)^2}{50\left(25 + \left(-10 - 5\sqrt{3}\right)^2\right)}\right]$

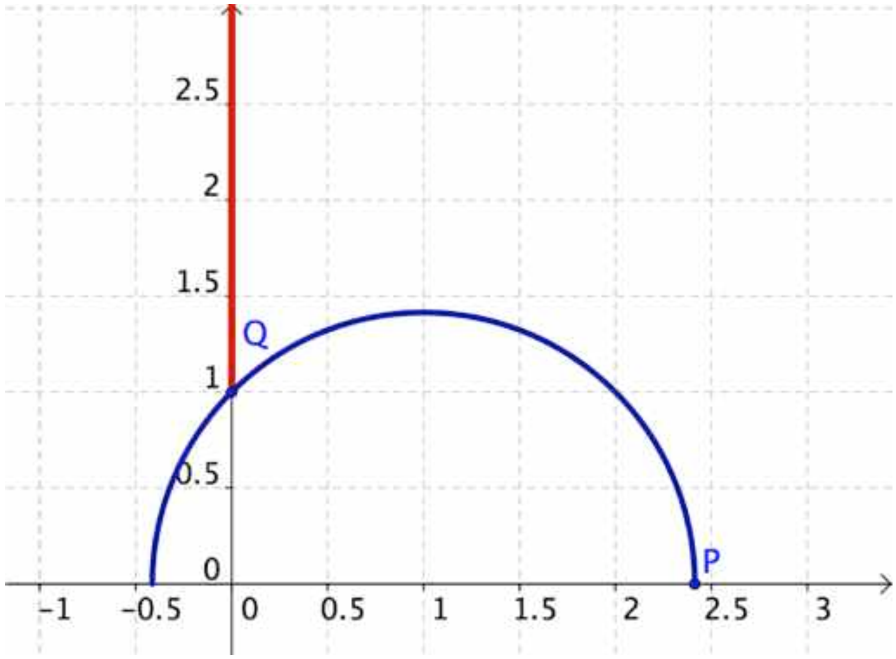
Which simplifies to

```
In[75]:= Simplify[%]
```

Out[75]= $\frac{2\pi}{3}$

■ 10.5 #9

In this problem one of our lines is vertical:



Because one of the PDIs is $(\infty, 0)$ we have to use one of the alternate forms given in Theorem 7. Let's say that $a = (\infty, 0)$, in which case we use the formula:

```
In[76]:= angle[g_, h_, u_] = ArcCos[((u - g) ^2 - h^2) / ((u - g) ^2 + h^2)]
```

Out[76]= $\text{ArcCos}\left[\frac{-h^2 + (-g + u)^2}{h^2 + (-g + u)^2}\right]$

And our answer is

```
In[77]:= angle[0, 1, 1 + Sqrt[2]]
```

Out[77]= $\text{ArcCos}\left[\frac{-1 + \left(1 + \sqrt{2}\right)^2}{1 + \left(1 + \sqrt{2}\right)^2}\right]$

Which simplifies to $\pi/4$ or 45 degrees:

```
In[78]:= angle[0, 1, 1 + Sqrt[2]] // FullSimplify
```

Out[78]= $\frac{\pi}{4}$