Problem 1: (8 Points) Let p and r be rays emanating from the origin with direction indicators $U = (1, 0)$ and $W = (-8, 15)$. Find a *unit* direction indicator V for the ray q which bisects $\angle(p, r)$.

When computing angle measures using our integral definition, we need to have unit direction indicators. So we begin by computing a unit vector in the direction of W:

$$W' = W/||W|| = (-8, 15)/\sqrt{(}64 + 225) = \left(\frac{-8}{17}, \frac{15}{17}\right)$$

Draw a picture to see that we want the angle formed by U and V to be congruent to the angle formed by V and W':

$$|\angle UOV| = |\angle VOW'|$$

$$\int_{\langle U,V\rangle}^{1} \frac{1}{\sqrt{1-t^2}}dt = \int_{\langle V,W'\rangle}^{1} \frac{1}{\sqrt{1-t^2}}dt$$

Hence we need to have $\langle U, V\rangle = \langle V, W'\rangle$. If we write $V = (v_1, v_2)$ that gives

$$v_1 = \frac{-8}{17}v_1 + \frac{15}{17}v_2$$

Solving for v_2 gives $v_2 = \frac{5}{3}v_2$. To find actual values for v_1 and v_2, we need to use the fact that V should be a unit vector, so that $v_1^2 + v_2^2 = 1$:

$$\begin{aligned} v_1 &= \frac{3}{5}v_2 \\ v_1^2 &= \frac{9}{25}v_2^2 \\ v_1^2 &= \frac{9}{25}(1 - v_1^2) \end{aligned}$$

This quadratic equation leads to $v_1 = \pm 3/\sqrt{34}$. The two possibilities for $V = (v_1, v_2) = (v_1, 5v_2/3)$ would be

$$(-3/\sqrt{34}, -5/\sqrt{34}) \text{ or } (3/\sqrt{34}, 5/\sqrt{34})$$

From a picture of the angle it's clear we need V to have a positive second component–i.e. it points up, not down, which you can also prove rigorously using Theorem 1.38 or other ideas. Hence $V = (3/\sqrt{34}, 5/\sqrt{34})$ is our final answer.

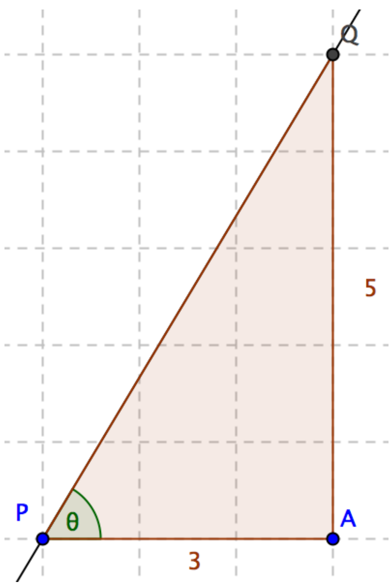
Problem 2: (8 Points) Construct the matrix formula for the the reflection across the line

$$\ell = \{(4, 2) + t(3, 5) : t \in \mathbb{R}\}.$$

Using the techniques developed in class, we know

$$\mathcal{M}_\ell(X) = F_\theta(X - P) + P$$

where P is a point on the line, $F_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$ is our orthogonal "reflection" matrix, and θ is the angle formed by ℓ with a horizontal line. By drawing the line and labeling θ, we can find the values for the matrix.



$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\frac{3}{\sqrt{34}}\right)^2 - \left(\frac{5}{\sqrt{34}}\right)^2$$
$$= \frac{9-25}{34} = -16/34 = -8/17$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\frac{3}{\sqrt{34}}\frac{5}{\sqrt{34}} = 30/34 = 15/17$$

Hence

$$\mathcal{M}_\ell(X) = \begin{bmatrix} -8/17 & 15/17 \\ 15/17 & 8/17 \end{bmatrix}\left(X - \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Problem 3: (8 Points) Construct the matrix formula for the isometry which sends $\angle(5,2)(2,2)(4,4)$ to $\angle(-1,4)(-1,1)(-3,3)$.

Most people who drew a careful picture immediately saw/guessed/conjectured that a rotation of $90^\circ = \pi/2$ was involved. The general form of a rotation centered at C is $R_\theta(X - C) + C$, where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is our rotation matrix. However, it's difficult to tell what C is in this case. There are many possible approaches. Here are two:

(1) Rotate by $\pi/2$ about the point $(2,2)$ using the steps developed in class. [Translate by $-(2,2)$, multiply by $R_{\pi/2}$, and translate back.] Then translate by $(-3,-1)$ to place the points on top of $\angle(-1,4)(-1,1)(-3,3)$:

$$\mathcal{U}(X) = \left(R_{\pi/2}\left(X - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$
$$= \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(X - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

(2) A simpler way is to move the original points to the origin [translate by $(-2,-2)$], rotate by $\pi/2$ [multiply by $R_{\pi/2}$] and then translate them directly to the correct location:

$$\mathcal{U}(X) = R_{\pi/2}\left(X - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(X - \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If you simplify the first formula you'll find it matches the second perfectly.[1] We're almost done, but there's one more important step. In problem 2 you were asked to construct a formula for a reflection, and you could just cite the formula from class. In this problem you had to construct an isometry which sends one thing to another, which means you need to justify your choice. It's not enough to say "it looks like a rotation followed by a translation." One way to justify your answer is to simply plug in the original points and show that it works:

$$\mathcal{U}(5,2) = (-1,4), \qquad \mathcal{U}(2,2) = (-1,1), \qquad \mathcal{U}(4,4) = (-3,3)$$

1.6.26: (6 Points)

Name the points $P = (123, 328)$, $Q = (17, -5)$, $R = (-96, -360)$ and $Y = (38, 61)$ to match the naming convention of Theorem 38. There are many ways to solve this problem. One is to note that there are unique constants a and c for which

$$Y - Q = a(P - Q) + c(R - Q)$$
$$(21, 66) = a(106, 333) + c(-113, -355)$$

This gives a linear system of two equations and two unknowns for which the solution is $a = -3$, $c = -3$. Part (iv) of Theorem 38 says Y is in the interior of $\angle PQR$ if and only if $a, c > 0$, so we conclude Y is *not* in the interior.

Another possible solution uses the first part of Theorem 38 and the "test for sided-ness," i.e. Proposition 16. $U = P - Q = (106, 333)$ is a direction indicator for the line through $\overleftrightarrow{PQ}$, so $A = (-333, 106)$ would be a coefficient vector, and

$$\langle A, X - P\rangle = 0$$

is a normal form of the line. You can check that

$$\langle A, Y - P\rangle = 3 > 0$$
$$\langle A, R - P\rangle = -1 < 0$$

By Proposition 16, Y and R must be on different sides of the line $\overleftrightarrow{PQ}$. Again by Theorem 38, we can conclude that Y is not in the interior of the angle. (You can do similar work to show that Y and P are on opposite sides of $\overleftrightarrow{QR}$, leading to the same conclusion.)

2.5.19: (7 Points)

[1] As it happens, if you multiply everything out you'll find they're both equal to

$$\mathcal{V}(X) = \mathcal{U}(X) = R_{\pi/2}\left(X - \begin{bmatrix}1\\0\end{bmatrix}\right) + \begin{bmatrix}1\\0\end{bmatrix}.$$

In other words, the isometry is a rotation of $\pi/2$ centered at the point $(1, 0)$. Some people were able to figure that out from the picture and used this simpler formula, which is fine!

By definition,

$$|\angle(C+U)C(C+V)| = \int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}}$$
$$|\angle(C+U)C(C+Z)| = \int_{\langle U,Z\rangle}^{1} \frac{dt}{\sqrt{1-t^2}}$$
$$|\angle(C+V)C(C+Z)| = \int_{\langle V,Z\rangle}^{1} \frac{dt}{\sqrt{1-t^2}}$$

By Proposition 9, we know that $|\angle(C+U)C(C+V)| = |\angle(C+U)C(C+Z)| + |\angle(C+V)C(C+Z)|$. Combining all of these facts, together with some calculus,

$$\begin{aligned}
|\angle(C+V)C(C+Z)| &= |\angle(C+U)C(C+V)| - |\angle(C+U)C(C+Z)| \\
&= \int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}} - \int_{\langle U,Z\rangle}^{1} \frac{dt}{\sqrt{1-t^2}} \\
&= \int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}} + \int_{1}^{\langle U,Z\rangle} \frac{dt}{\sqrt{1-t^2}} \\
&= \int_{\langle U,V\rangle}^{\langle U,Z\rangle} \frac{dt}{\sqrt{1-t^2}}
\end{aligned}$$

3.6.6: (6 Points)

(i) and (ii) are orthogonal; their columns are unit vectors and the first column is orthogonal to the second. Because they are orthogonal, you can simply transpose the matrices to find their inverses. (iii) is not orthogonal; its columns are not unit vectors, although they are close: they have length $\sqrt{290/289} \approx 1.00173$.

3.6.12: (7 Points)

Nearly any choice of a, b, P and Q with an isometry involving a translation will work as a counterexample. For example, let $\mathcal{U}(X) = X + (1,0)$ and choose $P = (0,0)$, $Q = (0,1)$, $a = 1$ and $b = 1$. Then $a + b = 2 \neq 1$ and

$$\mathcal{U}(aP + bQ) = \mathcal{U}(P + Q) = \mathcal{U}(0,1) = (1,1),$$
$$a\mathcal{U}(P) + b\mathcal{U}(Q) = (P + (1,0)) + (Q + (1,0)) = (1,0) + (1,1) = (1,2)$$