7.2.28: The formula for a central inversion centered at a point C is $\mathcal{C}_C(X) = -X + 2C$. Assume we have a line ℓ which includes C and forms an angle of θ with the horizontal; then the formula for reflection across ℓ is $\mathcal{M}_\ell(X) = F_\theta(X - C) + C$, where F_θ is the reflection matrix we've used consistently in class since October. Computing the two compositions:

$$\mathcal{C}_C \circ \mathcal{M}_\ell(X) = -(F_\theta(X - C) + C) + 2C = \cdots = -F_\theta(X - C) + C$$
$$\mathcal{M}_\ell \circ \mathcal{C}_C(X) = F_\theta((-X + 2C) - C) + C = \cdots = -F_\theta(X - C) + C$$

So each composition yields the same result. We now have to identify the net effect of this formula. Experimenting with GeoGebra (or other methods) suggests that the net result is reflection across a line through C which is perpendicular to ℓ, i.e. a line that forms an angle of $\phi = \theta + \pi/2$ with the horizontal. So we can check:median

$$F_\phi = F_{\theta+\pi/2} = \begin{bmatrix} \cos 2(\theta + \pi/2) & \sin 2(\theta + \pi/2) \\ \sin 2(\theta + \pi/2) & -\cos 2(\theta + \pi/2) \end{bmatrix}$$

Depending on how exactly you defined ϕ in terms of θ (you could also use $\phi = \theta - \pi/2$) you need to do some trig work similar to:

$$\cos 2(\theta + \pi/2) = \cos^2(2\theta + \sin^2 \pi) = \cos 2\theta \cos \pi - \sin 2\theta \sin \pi = -\cos 2\theta + 0$$
$$\sin 2(\theta + \pi/2) = \sin 2\theta + \pi = \sin 2\theta \cos \pi + \sin \pi \cos \theta = -\sin 2\theta + 0$$

Thus we can conclude

$$F_\phi = F_{\theta+\pi/2} = \begin{bmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix} = -F_\theta$$

Which means $\mathcal{C}_C \circ \mathcal{M}_\ell(X) = \mathcal{M}_\ell \circ \mathcal{C}_C(X) = -F_\theta(X - C) + C = F_\phi(X - C) + C$, so our conjecture above is correct: the compositions give a reflection across a line through C which is perpendicular to ℓ.

7.9.41: This problem has a lot of calculations which simplify to very nice answers. If $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is our standard rotation matrix, then the two given isometries are:

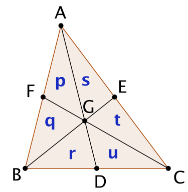
$$\mathcal{R}_1(X) = R_{2\pi/3}\left(X - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\mathcal{R}_2(X) = R_{4\pi/3}\left(X - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

If you crank out the algbera, you find

$$\mathcal{R}_1 \circ \mathcal{R}_2(X) = X + \begin{bmatrix} \sqrt{3} \\ 3 \end{bmatrix}$$
$$\mathcal{R}_2 \circ \mathcal{R}_1(X) = X + \begin{bmatrix} \sqrt{3} \\ -3 \end{bmatrix}$$

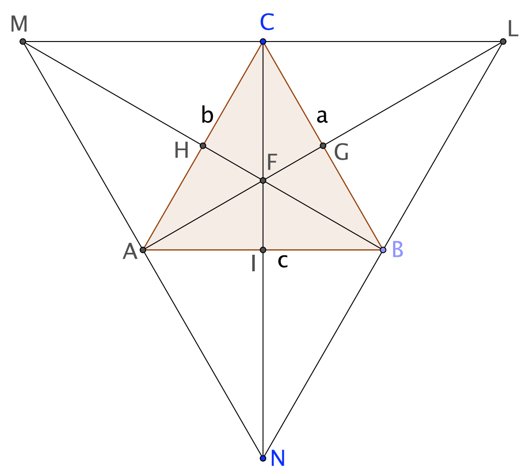
which represent translations by $(\sqrt{3}, 3)$ and $(\sqrt{3}, -3)$, respectively.

4.11.31: Consider the following picture, where D, E and F are the midpoints of the sides. I find it really hard to follow descriptions of all the different triangle areas (like $||\triangle GDC||$, $||\triangle ADB||$, and so on), so I've labeled each of the six small inner triangles with a blue letter which represents its area.



Following the hints given in class, $||\triangle ADB|| = ||\triangle ADC||$, so $p + q + r = s + t + u = \frac{1}{2}||\triangle ABC||$. If we can prove $p = q = r$, then each will be one third of one half of $||\triangle ABC||$, i.e. one sixth of $||\triangle ABC||$. To do so, first notice triangles p and q have congruent bases, $\overline{AF}$ and $\overline{FB}$, and equal heights, so $p = q$. Also, from the other hint in class, $p + q = 2r$. (Ask me to explain these hints if you don't remember them or if you weren't in class that day.) But then $2r = p + q = 2p$ and $p = q$. Or $2r = p + q = 2q$ gives $q = r$. So $p = q = r$, and a similar argument on the right shows that $s = t = u$. Hence each of the small triangles has area equal to $\frac{1}{6}||\triangle ABC||$.

4.11.44: From previous problems we know that, if a triangle is equilateral, then the centroid, incenter and orthocenter are all the same; because the altitudes are also the perpendicular bisectors of the sides, we can toss the circumcenter into that list as well. In this problem we prove the Fermat Point belongs in that list, too. The following picture shows the construction of the Fermat Point F for a triangle $\triangle ABC$, as described in section 4.9.



Suppose we know that the Fermat Point F happens to be the centroid, and we wish to prove that $\triangle ABC$ is equilateral. We can use the diagram above, except we can't assume $a = b = c$ or anything equivalent – that's what we're trying to prove! We know that the exterior triangles are equilateral, but we don't yet know that they're the same size.

Here's what we do know. Since F is the centroid, $\overline{AG}$ is a median and $|\overline{GC}| = |\overline{GB}|$. Hence (by SSS in Chapter 3) we know $\triangle LGC \cong \triangle LGB$. But then $|\angle LGC| = |\angle LGB|$ and these angles add to form a straight angle, meaning each of them is $\pi/2$. Hence $|\angle FGC| = |\angle FGB| = \pi/2$ as well, and $\overline{AG}$ is not only a median but also an altitude and perpendicular bisector! By SAS, $\triangle AGB \cong \triangle AGC$, which means $b = c$. Repeating this work with the other exterior triangles gives $a = b$ and $a = c$. Hence $a = b = c$ and the triangle is equilateral.

Conversely, suppose $\triangle ABC$ is equilateral and let F be the centroid. By previous problems it's also the incenter and orthocenter, and $\overline{AG}$, $\overline{BH}$ and $\overline{CI}$ are medians, angle bisectors, and altitudes. (For this part of the problem, ignore L, M, N, and any of the segments which are outside of the original triangle.) Using ASA or other congruence theorems, you can quickly show that all six of the little triangles formed inside $\triangle ABC$ are congruent. That means each of the six angles surrounding the centroid F are congruent, so each one is $360°/6 = 60°$. But then

$$|\angle AFC| = |\angle CFB| = |\angle BFA| = 120° = 2\pi/3$$

But forming those angles of 120° proves that the centroid F is the Fermat *minimizer* and therefore, by Theorem 28, the Fermat Point.

(There are many other possible proofs.)

5.3.22: In general, people broke this problem up into too many cases. We can prove $a \leq b \leq c \Leftrightarrow \alpha \leq \beta \leq \gamma$ in two steps. First, show that $a \leq b \leq c \Leftrightarrow \sin\alpha \leq \sin\beta \leq \sin\gamma$. This is almost immediate from the Law of Sines and nearly everybody handled it. (But ask me if you got stuck or have questions.) Then we need to prove that $\alpha \leq \beta \leq \gamma \Leftrightarrow \sin\alpha \leq \sin\beta \leq \sin\gamma$. Here, and *only* here, is where you need to worry about obtuse angles. (This second part wouldn't be true in general, except with the knowledge that α, β and γ are the angles of a triangle.) Here's the proof of that second part:

First assume $\alpha \leq \beta \leq \gamma$ and all three are less than or equal to $\pi/2$. On the interval $[0, \pi/2]$, sin is an increasing function. From calculus, that means $\alpha \leq \beta \leq \gamma \Leftrightarrow \sin\alpha \leq \sin\beta \leq \sin\gamma$.

Now assume $\alpha \leq \beta \leq \gamma$ and $\gamma > \pi/2$. For convenience, let $\theta = \pi - \gamma$. There are two important observations:

- $\sin\gamma = \sin\theta$. (Draw a picture and/or ask me if you're not sure why.)
- Because $\alpha+\beta+\gamma = \pi$, we have $\alpha+\beta = \theta$ which means α and β are smaller than both γ and $\theta = \pi-\gamma$. Furthermore, θ is less than $\pi/2$.

By the first part of our proof, $\alpha \leq \beta \leq \theta \Leftrightarrow \sin\alpha \leq \sin\beta \leq \sin\theta$. Since $\theta < \gamma$ and $\sin\theta = \sin\gamma$, this gives us $\alpha \leq \beta \leq \gamma \Leftrightarrow \sin\alpha \leq \sin\beta \leq \sin\gamma$.

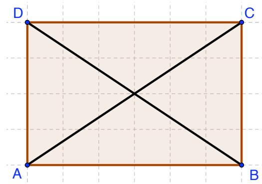
6.6.22: Let $PQRS$ be the vertices of a convex quadrilateral. Then the midpoints are $A = \frac{P+Q}{2}$, $B = \frac{Q+R}{2}$, $C = \frac{R+S}{2}$ and $D = \frac{S+P}{2}$. The vectors representing the sides are:

$$\begin{aligned} U = B - A &= \frac{Q+R-P-Q}{2} = \frac{R-P}{2} \\ V = C - B &= \frac{R+S-Q-R}{2} = \frac{S-Q}{2} \\ W = D - C &= \frac{S+P-R-S}{2} = \frac{P-R}{2} \\ X = A - D &= \frac{P+Q-S-P}{2} = \frac{Q-S}{2} \end{aligned}$$

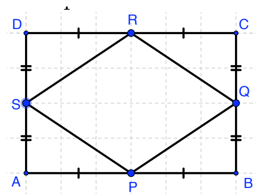
(Draw a picture and label all these points and vectors if this doesn't make sense!) Because $U = -W$ and $V = -X$, we see that $U \parallel W$ and $V \parallel X$. Hence $ABCD$ is a parallelogram.

6.6.24: The fastest way to complete this problem is a "cycle" of proofs like $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$, which amounts to three proofs. Unfortunately in this case it seems hard to make a connection between (ii) and (iii), so it's probably faster to prove $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$, a total of four proofs. I'll give **brief**

outlines of the proofs below. Regardless of which condition from (i), (ii) or (iii) we're assuming, we always know that $ABCD$ in the following picture is a parallelogram, so the opposite sides are always parallel and congruent, by Proposition 3.



- $(i) \Leftarrow (ii)$**:** Assume $ABCD$ is a rectangle, so all four vertex angles are congruent. Because it's also a parallelogram, $\overline{AD} \cong \overline{BC}$. Hence SAS tells us that $\triangle ABD \cong \triangle BAC$; in particular $\overline{AC} \cong \overline{BD}$, proving (ii).
- $(ii) \Leftarrow (i)$**:** Now assume $\overline{AC} \cong \overline{BD}$. Since opposite sides of a parallelogram are congruent, SSS tells us $\triangle ABD \cong \triangle BAC$; in particular $|\angle A| = |\angle B|$. You can use similar reasoning to show $|\angle B| = |\angle C|$ and $|\angle C| = |\angle D|$, which proves (i).
- $(i) \Leftarrow (ii)$**:** Assume $ABCD$ is a rectangle, so all four vertex angles are congruent. Because it's also a parallelogram, opposite sides are congruent. If you insert the midpoints of the sides, you can label congruent segments as follows:

  SAS tells us that the four small triangles are congruent, which means the four segments connecting P, Q, R and S are congruent. That means $PQRS$ is a rhombus, proving (iii).
- $(iii) \Leftarrow (i)$**:** Assume $PQRS$ in the above picture is a rhombus. Then all four triangles are congruent by SSS. That means the angles at A, B, C and D are congruent, proving (i).