You can make the equations in #38 much nicer by assuming that ℓ is the (top half of the) unit circle. Here's a sketch of how to do that, with plenty of details left for you to fill in. I would **highly recommend** you have scratch paper and pencil in hand while you read through this, and draw lots of pictures to understand what's going on.

(1) First, deal with the case where ℓ and m are both vertical lines.

(2) For the rest of the problem at least one, perhaps both, of ℓ and m are of the circular variety. So we can assume ℓ is of the form $(x-\omega)^2+y^2=\rho^2$; if not, then m would be circular, and we could just flip-flop the names.

(3) Translate the entire plane horizontally by $-\omega$; now ℓ has been translated until it's centered at the origin. Now dilate the whole plane by a factor of $1/\rho$ and explain why ℓ has now been sent to the unit circle – call it ℓ'. m has been sent to some other line m', which must be ultraparallel with ℓ'. (And you should explain why!) Let's suppose m' is directed by $(u,0)$ and $(w,0)$.

(4) Suppose n' is directed by $(p,0)$ and $(q,0)$ and is perpendicular to ℓ', i.e. the unit circle. You can find a very nice relationship between p and q. (Specifically, you can express q entirely in terms of p.) One way would be to use the fact that $[-1,1|p,q]=-1$ via Corollary 9. (In class we figured out the relationship you need here.)

(5) Once you know the directions indicators of n', you can use Corollary 9 again to say that n' is also perpendicular to m' if and only if $[u,w|p,q]=-1$. Since you can write q in terms of p, you really have just three variables in this equation. You want to solve for p [because you're trying to find the line n'] and your goal is to show this is only possible when m', the line directed by $(u,0)$ and $(w,0)$, is ultraparallel to ℓ', the unit circle. (I'll leave it to you to state what that means for the possible values of u and w.)

(6) The book refers to a certain messy expression which factors nicely. When I did this problem the tricky bit of factoring was this:

$$-4(u+w)^2+(-2-2uw)^2=4\left(-1+u^2\right)\left(-1+w^2\right)$$

(If you do the problem differently, however, you might never see this expression.)

(7) I'll give partial credit to anybody who can solve the problem under the assumption that one of the lines is the unit circle, as described here. For full credit you need to explain what you would do with the line n', which is perpendicular to ℓ' and m', in order to find a line n which is perpendicular to your original ℓ and m.

You may use the fact that horizontal translation and dilation of the plane are both isometries of the Poincare Half Plane, and all that entails.

Because I've sketched out a solution here I'll pay careful attention to clarity of writing and explanations. In other words: are you copying down these steps and adding a few words here and there, or do you really understand what you're doing and why this solution works? And did you solve the entire problem?

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