The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

1.6.58: Following the hints, and using the fact that $||U|| = ||V|| = 1$,

$$\begin{aligned} ||U+\alpha V||^2 &= \langle U+\alpha V, U+\alpha V\rangle \\ &= \langle U,U\rangle + \langle \alpha V, \alpha V\rangle + 2\langle U, \alpha V\rangle \\ &= ||U||^2 + \alpha^2||V||^2 + 2\alpha\langle U,V\rangle \\ &= 1+\alpha^2+2\alpha\langle U,V\rangle. \end{aligned}$$

At this point you could compute $||V+\alpha U||^2$ and discover that it's equal to the same expression. Alternatively, you could continue from above:

$$\begin{aligned} ||U+\alpha V||^2 &= 1+\alpha^2+2\alpha\langle U,V\rangle \\ &= ||V||^2 + \alpha^2||U||^2 + 2\alpha\langle V,U\rangle \\ &= \langle V,V\rangle + \langle \alpha U, \alpha U\rangle + 2\langle V, \alpha U\rangle \\ &= \langle V+\alpha U, V+\alpha U\rangle \\ &= ||V+\alpha U||^2. \end{aligned}$$

2.5.9: Let $P = (3,2)$, $Q = (2,-1)$ and $R = (4,5)$. As it happens, the rays $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are equal, meaning the angle has a measure of 0; this is evident if you draw a picture and you can prove it by noticing that $R = Q + 2(P-Q)$, where $P-Q$ serves as a direction indicator for the ray. Still, it can be instructive to go through the steps we'd normally take to measure an angle $\angle PQR$:

(1) Find a unit vector U which is a direction indicator for $\overrightarrow{QP}$:

$$U = \frac{P-Q}{||P-Q||} = \frac{(1,3)}{\sqrt{1+9}} = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

.

(2) Find a unit vector V which is a direction indicator for $\overrightarrow{QR}$:

$$V = \frac{R-Q}{||R-Q||} = \frac{(2,6)}{\sqrt{4+36}} = \left(\frac{2}{\sqrt{40}}, \frac{6}{\sqrt{40}}\right) = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

.

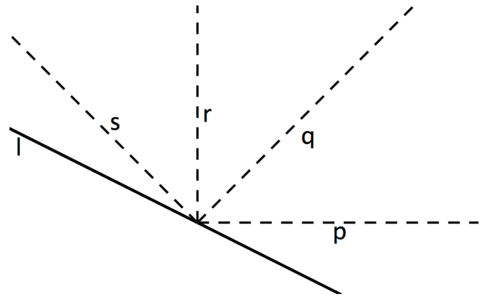
(3) Find the dot product of U and V:

$$\begin{aligned} \langle U,V\rangle &= \left\langle \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right), \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\right\rangle \\ &= \frac{1}{10} + \frac{9}{10} \\ &= 1 \end{aligned}$$

(4) Find the measure of $\angle PQR$:

$$|\angle PQR| = \int_{\langle U,V\rangle}^1 \frac{dt}{\sqrt{1-t^2}} = \int_1^1 \frac{dt}{\sqrt{1-t^2}} = 0$$

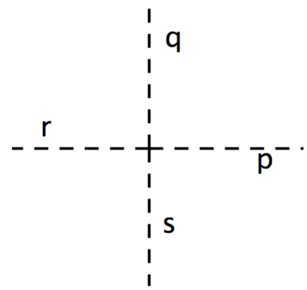
2.5.13: The situation looks something like this:



We can assume Proposition 9, but have to be careful about only applying it to two angles at once, each of which must meet along a common ray in the interior of the outer rays. Then

$$\begin{aligned}|\angle(p,s)| &= |\angle(p,q)| + (|\angle(q,s)|)\\ &= |\angle(p,q)| + (|\angle(q,r)| + |\angle(r,s)|)\end{aligned}$$

2.5.14: There are many possible answers. Here's one. Let p be the positive x-axis, q the positive y-axis, r the negative x-axis and s the negative y-axis:



With our definitions, $|\angle(p,s)| = \pi/2$, but:

$$|\angle(p,q)| + |\angle(q,r)| + |\angle(r,s)| = \pi/2 + \pi 2 + \pi 2 = 3\pi/2.$$

2.5.18: This problem was tricky, and the whole concept deserves a bit of an explanation. I've run through this concept multiple times with people in office hours, and it's easier to do in person than on paper. So if you read through the following solution and are still confused, please talk to me.

I assigned this problem because it helps demonstrate a quirk of our definition of angle measure. Go back and look at the solution for 2.5.9 to refresh your memory: to find the measure of an angle, we need two *unit* vectors U and V which define the angle, and then the measure is defined as an integral:

$$\int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}}$$

That definition is great mathematically, but it's a problem from a practical point of view since we can't actually evaluate that integral! So we have to resort to sneakiness:

(1) We *define* π to be the measure of an angle for which $\langle U,V\rangle = -1$. That's the definition we used in class, although your book states this as part of Theorem 6 in Chapter 2. As it happens, any such U and V represent a straight angle, as we would expect. [You actually proved on the first homework assignment that $U = -V$ in this case.]

(2) In class, we asked what would happen if $\langle U,V\rangle = 0$. By comparing the resulting integral to the definition of π, we decided such an angle would have measure $\pi/2$. So that's why a right angle has measure $\pi/2$; see the discussion on the top of page 58.

(3) In Example 10, the authors pick U, V and W which happen to construct *two* angles which must combine to form one right angle. (Draw a picture of these vectors!) We know those angles have

equal measure, since $\langle U, V\rangle = \langle V, W\rangle$, and the measure of an angle is entirely determined by the dot product of its (unit) direction indicators. So if two equal angles add to $\pi/2$, they must each measure $\pi/4$.

Solution 1

Following the pattern of Example 10, you might try to pick U, V and W which form two angles which add up to $\pi/2$ such that $|\angle UOV|$ is twice as large as $|\angle VOW|$. Then one would have to be $\pi/3$ and the other $\pi/6$. Using our prior knowledge of calculus and 30-60-90 triangles, you can even say what V would be:

$$\begin{aligned} U &= (1,0) \\ V &= (1/2, \sqrt{3}/2) \\ W &= (0,1) \end{aligned}$$

The problem is that its very hard to show that one of those angles is twice the other, i.e.:

$$\begin{aligned} \int_{\langle U,V\rangle}^{1} \frac{dt}{\sqrt{1-t^2}} &= 2\int_{\langle V,W\rangle}^{1} \frac{dt}{\sqrt{1-t^2}} \\ \int_{1/2}^{1} \frac{dt}{\sqrt{1-t^2}} &= 2\int_{\sqrt{3}/2}^{1} \frac{dt}{\sqrt{1-t^2}} \end{aligned}$$

It turns out to be easier to pick four points which define three angles:

$$\begin{aligned} U &= (1,0) \\ V &= (\sqrt{3}/2, 1/2) \\ W &= (1/2, \sqrt{3}/2) \\ X &= (0,1) \end{aligned}$$

You can verify that $\langle U, V\rangle$, $\langle V, W\rangle$ and $\langle W, X\rangle$ are all equal, so the angles have equal measure. All three of them must add up to $\pi/2$ by problem 2.5.13. Hence each is one third of $\pi/2$ – also known as $\pi/6$. Then you can combine angles. $|\angle UOW|$ must be $2\pi/6 = \pi/3$, and $\langle U, W\rangle = 1/2$, which completes the first part of the problem:

$$\arccos(1/2) = \int_{1/2}^{1} \frac{dt}{\sqrt{1-t^2}} = \int_{\langle U,W\rangle}^{1} \frac{dt}{\sqrt{1-t^2}} = \pi/3$$

For the second part, I'd probably define $Y = (-1/2, \sqrt{3}/2)$, show that $\langle X, Y\rangle = \sqrt{3}/2 = \langle U, V\rangle$, which imples that $|\angle XOY| = |\angle UOV| = \pi/6$, and then use Proposition 9 to show that $|\angle UOY| = 2\pi/3$. Since $\langle U, Y\rangle = -1/2$, I'd be finished. (See if you can write down the final line of the solution based on the information here.)

Solution 2

The first solution above mirrors Example 10 fairly closely, working in Quadrant 1. However, it does more than required; notice that we figured out $\arccos\sqrt{3}/2 = \pi/6$, even though we weren't asked to do so. Another option is to use the following points:

$$\begin{aligned} U &= (1, 0) \\ V &= (1/2, \sqrt{3}/2) \\ W &= (-1/2, \sqrt{3}/2) \\ X &= (0, 1) \end{aligned}$$

Again, you can check that $\langle U, V\rangle = \langle V, W\rangle = \langle W, X\rangle = \sqrt{3}/2$, which means all three angles are equal. By problem 13, $|\angle UOV| + |\angle VOW| + |\angle WOX| = |\angle UOX| = \pi$, so each of them is $\pi/3$. That proves the first part:

$$\arccos(1/2) = \arccos\langle U, V\rangle = |\angle UOV| = \pi/3.$$

To prove the second part, note that $\langle U, W\rangle = -1/2$. Using proposition 9 (to add two angles, as opposed to the three angles in problem 13).

$$\arccos(-1/2) = \arccos\langle U, W\rangle = |\angle U, W| = |\angle U, V| + |\angle V, W| = \pi/3 + \pi/3 = 2\pi/3.$$

Extra Problem: (a) In the two-dimensional case, which suffices for our current needs,

$$\begin{aligned} \langle aU, bV\rangle &= \langle a(u_1, u_2), b(v_1, v_2)\rangle = \langle (au_1, au_2), (bv_1, bv_2)\rangle \\ &= au_1bv_1 + au_2bv_2 = (ab)u_1v_1 + (ab)u_2v_2 \\ &= ab\langle U, V\rangle \end{aligned}$$

(b) We have $U = A/||A||$ and $V = B/||B||$, so by part (b):

$$\langle U, V\rangle = \langle A/||A||, B/||B||\rangle = \frac{1}{||A||}\frac{1}{||B||}\langle U, V\rangle$$

(d) Let U and V be unit vectors in the same direction as A and B. Using part(c),

$$\theta = \arccos(\langle U, V\rangle) = \arccos\left(\frac{\langle A, B\rangle}{||A|| \cdot ||B||}\right)$$

Now, using the fact that arccos has an inverse function cos, take cos of both sides:

$$\cos\theta = \frac{\langle A, B\rangle}{||A|| \cdot ||B||}$$

And rearrange to get the final formula.