Math 5335: Geometry Fall 2010 Homework 3 Solutions

The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

3.6.10: If $(0,0) = (r,s,t)^{\triangle ABC}$ for these particular points, then

$$\begin{aligned}(0,0) &= rA + sB + tC\\ &= r(3,-2) + s(4,-2) + t(4,-6)\\ &= (3r+4s+4t, -2r-2s-6t).\end{aligned}$$

Overall this gives us two equations, together with a third ($r+s+t=1$) which comes from the definition of barycentric coordinates:

$$\begin{aligned}3r+4s+4t &= 0\\ -2r-2s-6t &= 0\\ r+s+t &= 1\end{aligned}$$

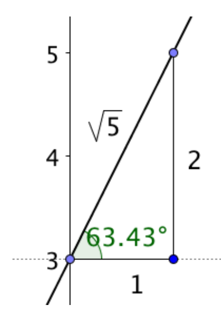
Solving this system gives $(0,0) = (4, -5/2, -1/2)^{\triangle ABC}$. Solving a similar system for the point $(4,5)$ gives

$$(4,5) = (0, 11/4, -7/4)^{\triangle ABC}$$

7.9.6,7: The translation in these problems is easy to write a formula for: $\mathcal{T}(X) = X + (-3,-6)$ or, if you prefer, $\mathcal{T}(x,y) = (x-3, y-6)$. The equation for the reflection is trickier. We know $\langle (2,-1),(x,y)\rangle = -3$ is equivalent to $2x - y = -3$, or $y = 2x+3$. So the slope of the mirror is 2, and hence the angle it forms with a horizontal line is $\theta = \arctan 2 \approx 63.4°$. The y-intercept form of the line makes it clear that $(0,3)$ is on the line. Hence a matrix formula for the reflection across this line is:

$$\mathcal{M}(X) = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\left([X] - \begin{bmatrix}0\\3\end{bmatrix}\right) + \begin{bmatrix}0\\3\end{bmatrix}$$

Here's a picture of the line:



Using the triangle in the picture, we see that

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}$$
$$\sin 2\theta = 2\cos\theta\sin\theta = \frac{4}{5}$$

Hence our formula for the reflection becomes

$$\mathcal{M}(X) = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \left([X] - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Now consider the two compositions (check the parentheses carefully to make sure you see the differences!):

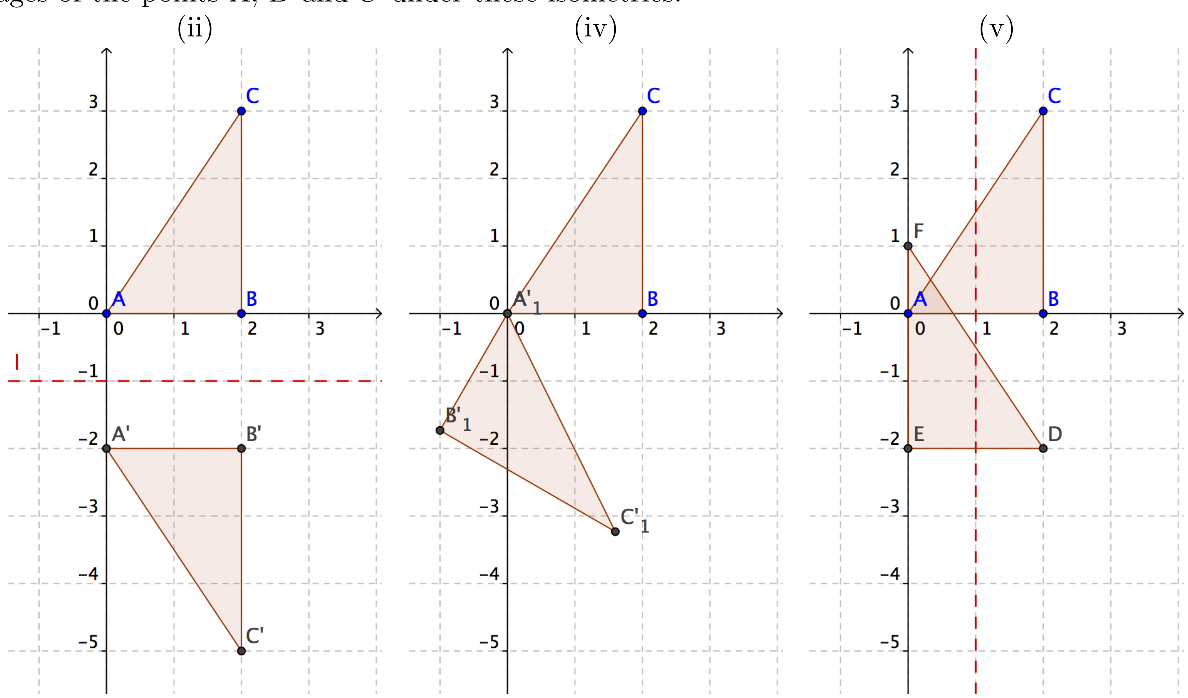
$$\mathcal{T} \circ \mathcal{M}(X) = \left(\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \left([X] - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} -3 \\ -6 \end{bmatrix}$$
$$\mathcal{M} \circ \mathcal{T}(X) = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \left(\left([X] + \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right) - \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

If you distribute across the parentheses, multiply the constant vectors by the matrix, and collect terms, you'll find that these are both (!) equal to

$$\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} [X] + \begin{bmatrix} -27/5 \\ -24/5 \end{bmatrix}$$

So with these particular choices, it doesn't matter if you do the reflection first and then the translation, or vice versa; your answers to #6 and #7 are the same. (Why does it turn out that way? Hint: $(-3,-6) = -3(1,2)$ could serve as a direction indicator for the line, so the composition in either order gives the same glide reflection!)

7.9.18: Refer to the illustrations below; ask me if you had difficulty computing any of the images of the points A, B and C under these isometries.



7.9.26: Let $\mathcal{C}_C(X) = -X + 2C$ be a central inversion about a center C, and $\mathcal{T}_P(X) = X + P$ be a translation by P. Then:

(i) $\mathcal{T}_P \circ \mathcal{C}_C(X) = \mathcal{T}_P(-X + 2C) = -X + 2C + P = -X + 2\left(C + \frac{1}{2}P\right)$ This is the formula for a central inversion centered at the point $\left(C + \frac{1}{2}P\right)$.

(ii) $\mathcal{C}_C \circ \mathcal{T}_P(X) = \mathcal{C}_C(X + P) = -X - P + 2C = -X + 2\left(C - \frac{1}{2}P\right)$ This is the formula for a central inversion centered at the point $\left(C - \frac{1}{2}P\right)$.

7.9.30: (i) The two lines are perpendicular and intersect at $C = \left(\frac{47}{68}, \frac{1}{68}\right)$. By our work in class, that means the composition of reflections across the two lines is a central inversion, centered at C:

$$\mathcal{C}_{(47/68,1/68)}(X) = -X + 2(47/68, 1/68)$$

(ii) The two lines intersect at $(2, 0)$ when $t = 0$ and $s = -1/2$. The angle α from the first line to the second is about $59.03°$; from our work in class, the net result is a rotation about the point $(2, 0)$ by an angle of 2α. As with earlier homework problems, we can draw triangles on these lines to determine exact values of $\cos 2\alpha$ and $\sin 2\alpha$ in order to write down our rotation matrix. The resulting matrix formula is:

$$\mathcal{R}(X) = \frac{1}{17}\begin{bmatrix} -8 & -15 \\ 15 & -8 \end{bmatrix}\left(X - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Or, if you multiply everything out,

$$\mathcal{R}(X) = \frac{1}{17}\begin{bmatrix} -8 & -15 \\ 15 & -8 \end{bmatrix} X + \begin{bmatrix} 50/17 \\ -30/17 \end{bmatrix}$$

(iv) The lines $j : (2, 5) + s(3, -3)$ and $k : \langle(-3, 3), X\rangle = 0$ are perpendicular and meet at $C = (7/2, 7/2)$, as you can verify. (Ask me if you're not sure how!) So as in (i), we have a central inversion at a point:

$$\mathcal{C}_C(X) = -X + 2(7/2, 7/2) = -X + (7, 7)$$