The following is a non-comprehensive list of solutions to homework problems. In some cases I may give an answer with just a few words of explanation. On other problems the stated solution may be complete. As always, feel free to ask if you are unsure of the appropriate level of details to include in your own work.

Please let me know if you spot any typos and I'll update things as soon as possible.

4.11.9: If $\triangle ABC$ is equilateral, then $a = b = c$, so the incenter (by Theorem 18) is

$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle ABC} = \left(\frac{a}{a+a+a}, \frac{a}{a+a+a}, \frac{a}{a+a+a}\right)^{\triangle ABC} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle ABC},$$

which is the centroid. Conversely, suppose the centroid and incenter are equal:

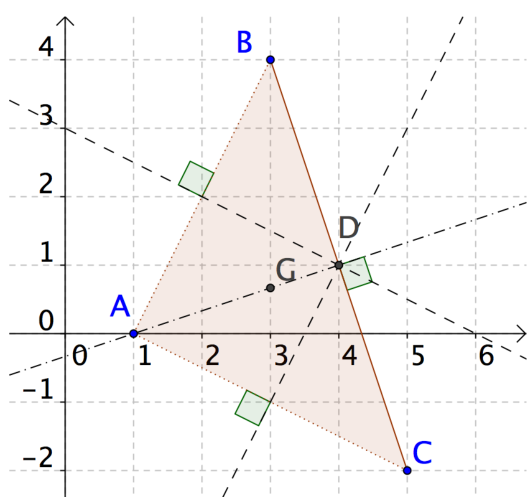
$$\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c}\right)^{\triangle ABC} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle ABC}$$

Equating the parts of the barycentric coordinates (and multiplying by 3) gives three equations:

$$3a = a + b + c$$
$$3b = a + b + c$$
$$3c = a + b + c$$

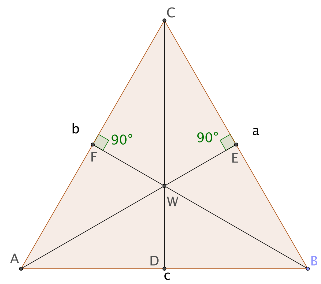
You can check that this system of equations is satifsfied if and only if $a = b = c$, so that the triangle is equilateral. (Come talk to me if you had trouble with this system.)

4.11.21: The picture below illustrates the work. The altitudes are represented with dotted lines, including two of the sides of the triangle. (This happens because it's a right triangle.) The dashed lines are perpendicular bisectors and meet at D, which is both the circumcenter and the foot of the altitude from A. Hence the line from A to D is both dashed and dotted. G is the centroid, and the orthocenter is at A. The Euler line passes through A, G and D. Its equation is $x - 3y = 1$ or, in normal form, $\langle (1, -3), X\rangle = 1$. A parametric representation is $A + t(3, 1)$ or $(1, 0) + t(3, 1)$.



4.11.24: If $\triangle ABC$ is equilateral, then $a = b = c$, so you can use the barycentric coordinates of the incenter and orthocenter to show these are both (after much simplication) equal to $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle}$.

In the other direction, we assume that the incenter equals the orthocenter and must show that the triangle is equilateral. One way is to set up a huge system of equations where the barycentric coordinates of the orthocenter are equal to the barycentric of the coordinates of the incenter, and show this is only true if $a = b = c$. That's possible but messy. I said in class you could use the ASA congruence theorem for triangles for this problem, even though it doesn't appear in Chapter 5. One possible solution follows:



Let W be the point which is both the incenter and orthocenter. Then W is incident with all angle bisectors and all altitudes of the triangle. Any line from a vertex through the incenter is an angle bisector, and any line from a vertex through the orthocenter is an altitude. Hence the angle bisectors are altitudes and vice versa. In particular, in the picture above $\overline{AE}$ is a bisector and altitude, so we can conclude:

$$|\angle WAC| = |\angle WAB|$$

$$|\angle AEC| = |\angle AEB| = \pi/2$$

Using $\overline{AE}$ as a side in each triangle, ASA then implies $\triangle AEC \cong \triangle AEB$, hence $b = c$. You can prove similarly that $\triangle BFC \cong \triangle BFA$ so that $a = c$. Thus $a = b = c$ and $\triangle ABC$ is equilateral.

5.3.9: Assume $\triangle ABC$ is equilateral, so $a = b = c$. Since $a = b$, Proposition 15 implies that $\alpha = \beta$. Similarly, $a = c$ gives us $\alpha = \gamma$. Hence $\alpha = \beta = \gamma$ and our triangle is equiangular.

In the other direction, assume $\alpha = \beta = \gamma$. Using Proposition 15 in the other direction, $\alpha = \beta$ implies $a = b$. And $\alpha = \gamma$ gives $a = c$. Thus $a = b = c$ and $\triangle ABC$ is equilateral.

5.3.18: In (i) and (ii) the inequalities of Proposition 9 are satisfied, and you can form triangles. Your answers will vary according to which vertices you choose. In (iii) to triangle is possible since $2+2 < 5$, so the triangle inequality doesn't hold.