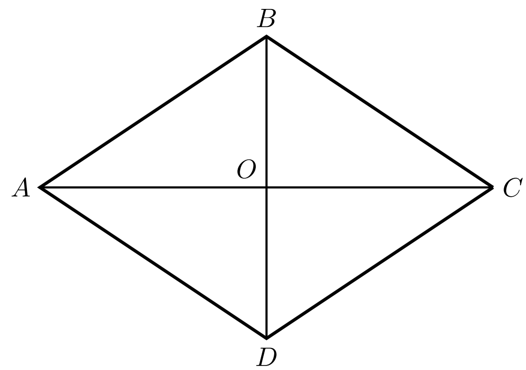
8.24: This problem requires a lot of proofs. Almost everybody used geometric diagrams to analyze angles, construct congruent triangles, etc. to complete their proofs. Those methods are perfectly valid, but just to demonstrate the power of our vector methods, I'll use those at times as well.

Most people proved $(i) \Leftrightarrow (ii)$ and $(i) \Leftrightarrow (iii)$ but other approaches are possible. I'll give sketches of proofs below, leaving you to fill in the details on the diagrams about which segments and angles are congruent, etc.

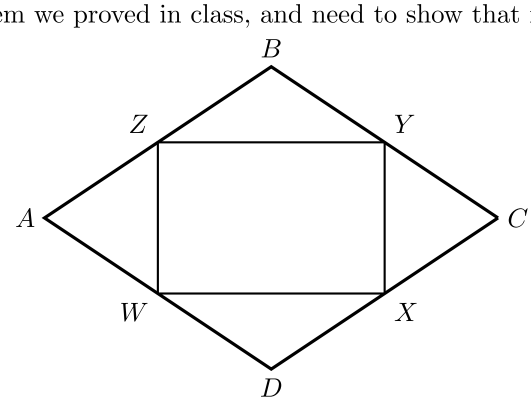
$(i) \Rightarrow (ii)$**:** Assume $ABCD$ is a rhombus and draw in the diagonals $\overline{AC}$ and $\overline{BD}$.



Because $ABCD$ is a parallelogram, its diagonals bisect each other, so $\overline{BO} \cong \overline{DO}$ and $\overline{AO} \cong \overline{CD}$. Along with the fact that all four sides of the rhombus are congruent, this is enough to say all four triangles ($\triangle AOB$, $\triangle COB$, $\triangle COD$ and $\triangle AOD$) in the diagram are congruent by SSS. Hence all four angles at O are congruent, via CPCTC, and their measures add up to 2π. Therefore each of them is $2\pi/4 = \pi/2$, so the diagonals are perpendicular.

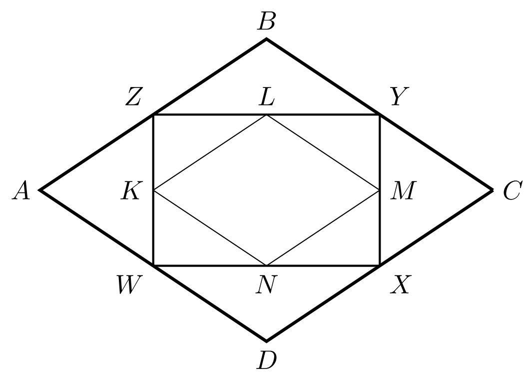
$(ii) \Rightarrow (i)$**:** We still know that the diagonals of $ABCD$ bisect each other, and now we assume the angles at O all measure $\pi/2$. Hence we can use SAS congruence to show all four triangles are congruent. In particular their hypoteni – the four sides of $ABCD$ – are congruent, so $ABCD$ is a rhombus.

$(i) \Rightarrow (iii)$**:** Given that $ABCD$ is a rhombus and W, X, Y and Z are midpoints of the sides, we know all eight line segments around the edge ($\overline{AZ}$, $\overline{ZB}$, $\overline{BY}$, etc.) are all congruent. We also know $WXYZ$ is a parallelogram, by the theorem we proved in class, and need to show that it's a rectangle.



Most people used congruent triangles and supplementary angles to show each of the angles in $WXYZ$ is a right angle. Here's a different approach, using vectors, Proposition 8.6 (which you proved on homework) and the fact that $(i) \Leftrightarrow (ii)$, which we proved above.

We know $W = \frac{A+D}{2}$, $X = \frac{C+D}{2}$, $Y = \frac{B+C}{2}$, and $Z = \frac{A+B}{2}$. By Proposition 8.6, $WXYZ$ is a rectangle if the midpoints of its sides form a rhombus:

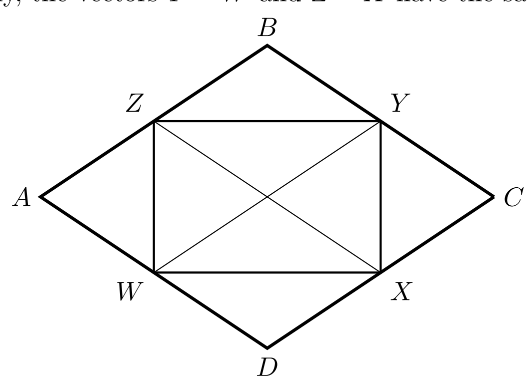


We know $K = \frac{W+Z}{2}$, $L = \frac{Y+Z}{2}$, $M = \frac{X+Y}{2}$, and $N = \frac{W+X}{2}$. You can do the simplifications to check:

$$N - L = \cdots = \frac{D-B}{2}$$
$$M - K = \cdots = \frac{C-A}{2}$$

Because $ABCD$ is a rhombus, $D - B \perp C - A$ by $(i) \Rightarrow (ii)$. Because $N - L$ and $M - K$ are scaled versions of those vectors, they are also perpendicular. By $(ii) \Rightarrow (i)$, that means $KLMN$ is a rhombus. Hence $WXYZ$ is a rectangle.

$(iii) \Rightarrow (i)$**:** In this last step, we assume $WXYZ$ is a rectangle, which means its diagonals $\overline{WY}$ and $\overline{XZ}$ are congruent. Put differently, the vectors $Y - W$ and $Z - X$ have the same length.



Now let's use our vector formulas for the midpoints, along with some facts about any parallelogram $ABCD$. We know $B - A = C - D$. Furthermore, using vector arithmetic, we see

$$Y - W = \cdots \frac{B-A}{2} + \frac{C-D}{2} = B - A \text{ (since } B - A = C - D\text{)}.$$

In other words, $\overline{WY} \cong \overline{AB} \cong \overline{DC}$. Similarly, $\overline{XZ} \cong \overline{AD} \cong \overline{BC}$. Because $\overline{WY} \cong \overline{XZ}$, that means all four sides of $ABCD$ are congruent.