Chapter 1 - Flythrough

This will be a fast review/intro – You're responsible for reading Chapter 1 (and 2) and talking to me if there's something you don't follow.

(Think: big ideas/terms, like "set," not the intricacies of, say, proof of Proposition 1.30...)

Fundamentals / Vocabulary

$\forall$: for all, for every

$\exists$: there exists ($\exists!$: there exists a unique)

iff : if and only iff, $\Longleftrightarrow$

We won't use sets in much depth. Mostly:

$\mathbb{R} = \{\text{real \#'s}\} = \{x : x \in \mathbb{R}\} = \{x \mid x \in \mathbb{R}\}$

$\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

and esp. subsets of those

Recall "Abstract" Function Notation

$f: A \to B$ A: domain, inputs

$x \mapsto f(x)$ B: codomain (range, image)

$x \rightsquigarrow f(x)$ set of possible outputs

Ex $f: \mathbb{R} \to \mathbb{R}$ or $\{x \in \mathbb{R} : x \geq 0\}$

$x \mapsto x^2$

$[\, f(x) = x^2 \,]$

Def f is injective (or one-to-one, $1:1$) if any two different inputs are sent to diff. outputs: $x \neq y \Rightarrow f(x) \neq f(y)$.

f is surjective (onto) if every elt of codomain is actual output:

$\forall\, b \in B \ \exists\, a$ such that $f(a) = b$.

Functions Sheet

	Injective		Surjective
1.	✗	$f(1) = f(3) = 4$	✓
2.	✓		✗
3.	✓		✗
4.	✓		✓
5.	✓		✗
6.	✓		✗
7.	✓		✗
8.	✗		✗
9.	✗		✗
10.		Not a Function	

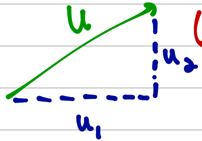
Vectors, Points and Lines

A (2D) vector is an ordered pair of real #'s, (a,b).

Common notations: $\langle a, b\rangle$, $\overrightarrow{(a,b)} = \vec{u}$

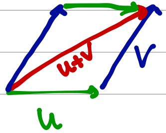
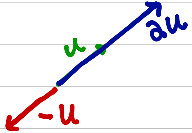
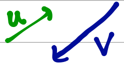
Our book: $U = (u_1, u_2)$ $X = (x_1, x_2)$

Graphically, U is an arrow:



(here $u_1, u_2 > 0$)

Virtually every vector concept has an algebraic def^n/ interpretation and a geometric/axiomatic one.

	alg	geo
Addition	$U+V=(u_1,u_2)+(v_1,v_2)$ $=(u_1+v_1,\ u_2+v_2)$	
(scalar) mult'n	$cU=c(u_1,u_2)$ $=(cu_1,\ cu_2)$	
Subtraction	$U-V=U+(-1)V$	
linearly dependent	$U=cV$ or $V=cU$ $\exists\, a,b$ such that $aU+bV=0$, a,b not both 0.	 (parallel)

alg | geo

$u \cdot v = \langle u, v \rangle$

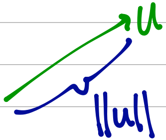
dot product: (inner product, scalar product)

$$u \cdot v = (u_1, u_2) \cdot (v_1, v_2)$$
$$= u_1 v_1 + u_2 v_2$$
$$\neq (u_1 v_1, u_2 v_2)$$

?

length/magnitude

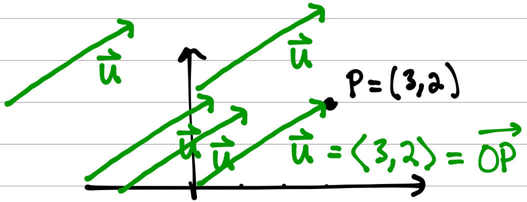
$$u \cdot u = (u_1, u_2) \cdot (u_1, u_2)$$
$$= u_1^2 + u_2^2$$
$$= \|u\|^2$$



$$\|u\| = \sqrt{u \cdot u}$$

⚠ If you learned vectors from Stewart's book...

Stewart make huge distinctions between points and vectors:

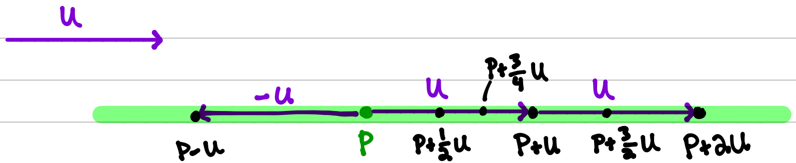


<u>We won't</u>. For us, vector and point are synonyms.

It's clear from context, and it makes life easier to do it this way. To wit:

Def Given a point P and non-zero vector u, the set

$\ell = \{P + su : s \in \mathbb{R}\}$ is a <u>line</u>.

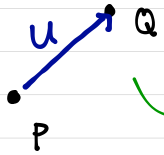


u is direction indicator (dir'n vector). Points on ℓ are incident with ℓ, and are collinear.

Ex $(-1,2)+s(3,-4)$ $\quad (\frac{1}{2},0)$ is on line $(s=\frac{1}{2})$

$(5,6)$ is not. $\left.\begin{aligned} -1+3s &= 5 \\ 2-4s &= 6 \end{aligned}\right\}$ no sol'n

⚠ Important Example. What's u?

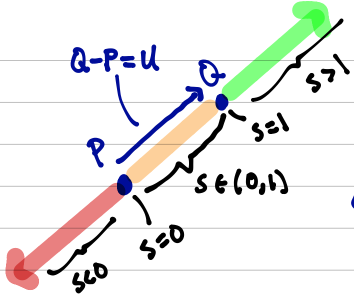


$$\underline{u} = \underline{Q-P}$$

want $P + u = Q$

$$u = Q - P$$

$P + s(Q-P)$



$$\overline{PQ} = \overline{QP}$$

$$\{P+s(Q-P)\} \qquad \{Q+s(P-Q)\}$$

$$\overleftrightarrow{PQ} = \overleftrightarrow{QP}$$

$$\overrightarrow{PQ} \neq \overrightarrow{QP}$$

$$\|X\| = \sqrt{X \cdot X}$$

length of $\overline{PQ}$ is $\|Q-P\| = \|P-Q\|$

$$= \sqrt{\langle Q-P,\ Q-P\rangle}$$

$$= \sqrt{(Q-P)\cdot(Q-P)}$$

day 1

Warmup Problem (9/12/18)

Recall: $U \cdot V = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$

Prove: The dot product is commutative: $U \cdot V = V \cdot U$

$$U \cdot V = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = V \cdot U$$

Prove: The dot product is distributive: $U \cdot (V + W) = U \cdot V + U \cdot W$

$$U \cdot (V + W) = (u_1, u_2) \cdot (v_1 + w_1, v_2 + w_2)$$

$$= u_1(v_1 + w_1) + u_2(v_2 + w_2)$$

$$= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2$$

$$= (u_1 v_1 + u_2 v_2) + (u_1 w_1 + u_2 w_2)$$

$$= U \cdot V + U \cdot W$$

Does the def^n $\{P + sU\}$ cover everything we expect?

- Can get segments, rays using restricted values of s.
- Two points form a line? Yes (wksheet)

<u>Prop 1.4</u> two non-zero vectors are DI's of same line <u>iff</u> they're scalar mult's of each other.

Let $P \neq Q$. Then $\exists$ unique line $\overleftrightarrow{PQ}$ incident with both, $U = Q - P$ is a DI of $\overleftrightarrow{PQ}$, and <u>every</u> DI of $\overleftrightarrow{PQ}$ is difference of two pts on the line.

Other Forms

Ex $(-1,2)+s(3,4) = (-1,2)+(3s,4s) = (\underbrace{-1+3s}_{x}, \underbrace{2+4s}_{y})$

$$\left.\begin{aligned} x &= 3s-1 \\ y &= -4s+2 \end{aligned}\right\} \Rightarrow \begin{aligned} s &= \tfrac{1}{3}(x+1) \\ s &= -\tfrac{1}{4}(y-2) \end{aligned}$$

$$\tfrac{1}{3}(x+1) = -\tfrac{1}{4}(y-2)$$

$$(y-2) = -\tfrac{4}{3}(x+1) \quad \text{(Pt slope)}$$

$$y = 2-\tfrac{4}{3}x-\tfrac{4}{3}$$

$$y = -\tfrac{4}{3}x+\tfrac{2}{3} \quad \text{(slope int.)}$$

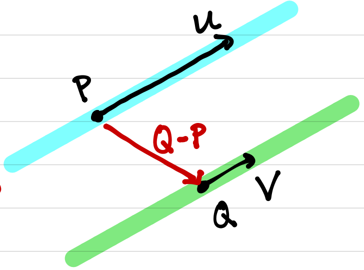
Def Two lines ℓ, m are parallel, $\ell \parallel m$, if their DI's are $\parallel$.

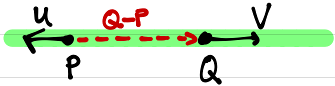
Prop 1.6 Lines $\ell = \{P + sU\}$, $m = \{Q + tV\}$

- $\cap$ in one pt if U, V linearly independent (not $\parallel$)

- empty $\cap$'n if $U \parallel V$ and $U \nparallel Q-P$

- same line if $U \parallel V$ and $U \parallel Q-P$



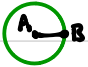


Quick Status Check: which of Euclid's Axioms work so far?

1. Given two pts, $\exists$ line containing them — yes

2. Lines can be extended indefinitely — yes

3. Given A, B, $\exists$ circle cent'd at A with radius $\overline{AB}$.



$r = \|B - A\|$

$C = \{X : \|X - A\| = r\}$

4. Right angles are all equal — ✗

5. $\parallel$ postulate — (yes, HW)

Perependicularity / Orthogonality

Def $U \perp V$ if $U \cdot V = 0$. Two lines are perpendicular if their DI's are $\perp$.

$\parallel$ and $\perp$ play important roles...

Corollary 1.11 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\parallel$ to ℓ.

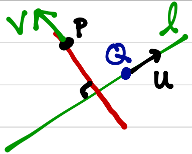
Prop 1.15 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\perp$ to ℓ.

Corollary 1.16 (Lemma) The set of vectors $\perp$ to $U=(u_1,u_2)\neq 0$ consists of all multiples of $(-u_2,u_1)$

Pf: First, we see that $(u_1,u_2)\cdot(-u_2,u_1) = -u_1u_2+u_2u_1=0$.

Now suppose $V \perp U$, so that $u_1v_1+u_2v_2=0$. We want to show $V=c(-u_2,u_1)$ for some c.

Prop 1.15 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\perp$ to ℓ.



Pf Suppose $\ell: Q + sU$, $U \neq 0$

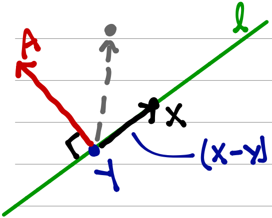
Then $m: P + s\underbrace{(-u_2, u_1)}_{V}$ is $\perp$ ℓ.

Suppose $\exists$ another such line $P + sW$, so $U \perp W$
By prev. slide $W = cV$ for some c, so $W \| V$.

By Prop 1.6, they are same line.

Normal Form

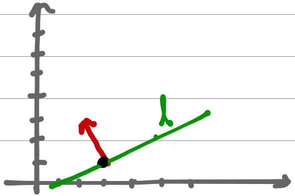
Given line ℓ, choose $Y \in \ell$ and $A \perp \ell$ (i.e. $A \perp U$, U <u>any</u> DI of ℓ). Then



$$\ell = \{ X : \underbrace{A \cdot (X-Y) = 0}_{\text{nl eqn of line}} \}$$

⚠ A, Y are constants; $X = (x_1, x_2)$
$(= (x, y))$

Ex $Y = (3,1)$, $A = (-1,2)$



$(-1,2) \cdot (X - (3,1)) = 0$

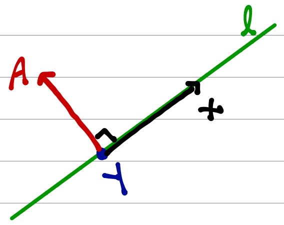
$(-1,2) \cdot ((x_1,x_2) - (3,1)) = 0$

$(-1,2) \cdot (x_1 - 3, x_2 - 1) = 0$

$-(x_1 - 3) + 2(x_2 - 1) = 0$

$(x_2 - 1) = \frac{1}{2}(x_1 - 3)$ $\quad$ $y = \frac{1}{2}x - \frac{1}{2}$

$\exists$ alternate version of normal form:



$$A \cdot (X - Y) = 0$$

$$A \cdot X - A \cdot Y = 0$$

$$A \cdot X = \underbrace{A \cdot Y}_{=c}$$

$$A \cdot X = c$$

Ex $A = (-1, 2)$, $Y = (3, 1)$

$$(-1,2) \cdot \big((x,y) - (3,1) \big) = 0$$

$$(-1,2) \cdot (x,y) - (-1,2) \cdot (3,1) = 0$$

$$(-1,2) \cdot (x,y) + 1 = 0 \qquad (-1,2) \cdot X = -1$$

Second Warmup Question:

Prove: $(cU)\cdot V = c(U\cdot V)$ $\quad (= V\cdot(cU)$ etc... $)$

Pf:
$$\begin{aligned}(cU)\cdot V &= (cu_1, cu_2)\cdot(v_1, v_2)\\ &= cu_1v_1 + cu_2v_2\\ &= c(u_1v_1 + u_2v_2)\\ &= c(U\cdot V)\end{aligned}$$

Prove: $\forall\ U,\ \|-U\| = \|U\|$

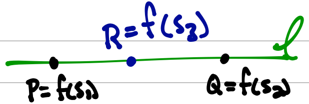
$\cancel{\sqrt{(-U)\cdot(-U)} = \sqrt{U\cdot U}}$
$\vdots$ (avoid)
$0 = 0$

$$\begin{aligned}\|-U\|^2 &= (-U)\cdot(-U)\\ &= [(-1)U]\cdot[(-1)U]\\ &= (-1)^2\, U\cdot U\\ &= U\cdot U\\ &= \|U\|^2\end{aligned}$$

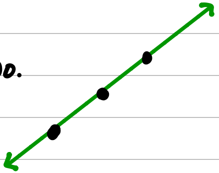
Betweeness

$/\!/ \; B+sU$

Def/Prop Let $f(s)$ be eqn of line, $f(s_1)=P$, $f(s_2)=Q$. Then R is between P, Q if $\exists\, s_3$, $s_1 < s_3 < s_2$, $f(s_3)=R$

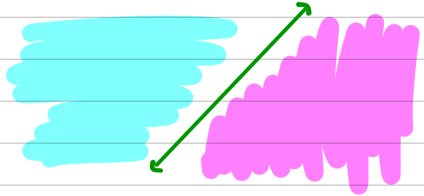


Corollary 1.22 Given 3 pts on a line, one must be b/w other two.

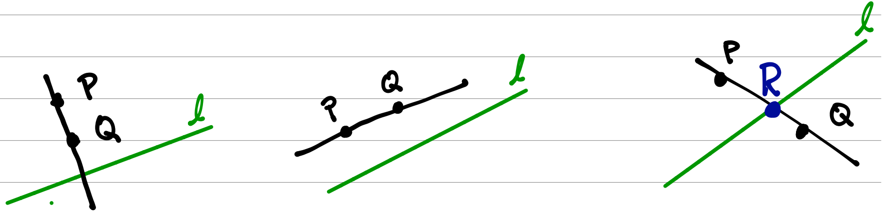


See book for further corollaries with 4+ points

A line separates $\mathbb{R}^2$ into two "half planes."



<u>Clever Def</u> $P, Q \notin \ell$ on opposite sides of ℓ if $\exists\ R \in \ell$ which is between them

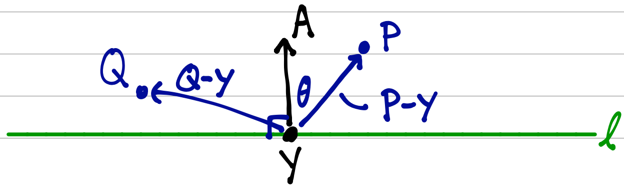


Prop 1.30 Let $\ell: A\cdot(X-Y)=0$ $(Y\in\ell,\ A\perp\ell)$ and $P, Q \notin \ell$. Then P and Q are on same/opposite side of ℓ if $A\cdot(P-Y)$, $A\cdot(Q-Y)$ have same signs.

⚠ Book uses $A\cdot X = c$, compares $A\cdot P - c$, $A\cdot Q - c$.

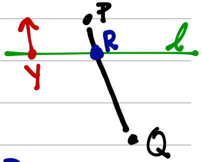
⚠‼ Would be "simple" (well, simpl<u>er</u>) if we had

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cos\theta \quad \begin{cases} > 0 & \text{for } \theta\in[0,\pi/2) \\ < 0 & \text{for } \theta\in(\pi/2,\pi] \end{cases}$$



Prop 1.30 Let $\ell: A\cdot(X-Y)=0$ $(Y\in\ell,\ A\perp\ell)$ and $P, Q\in\ell$. Then P and Q are on same/opposite side of ℓ if $A\cdot(P-Y)$, $A\cdot(Q-Y)$ have same/opposite sign.

Pf Let $g(s)=A\cdot\big[\underbrace{(P+s(Q-P))}_{\text{line seg. } \overline{PQ}}-Y\big]$ for $0\le s\le 1$



$g(s)=0$ for some s iff $\overline{PQ}$ intersects ℓ at pt R
($\Rightarrow$ P, Q opposite sides)

$$g(s)=\underbrace{A\cdot(P-Y)}_{\#}+s\,\underbrace{A\cdot(Q-P)}_{\#}=\cdots=b+sc \qquad (1+2s)$$

That's linear! min/max's at endpts, can be 0 if +/- (or -/+) at endpts

$$g(0)=A\cdot(P-Y)$$
$$g(1)=A\cdot(Q-Y)$$

Moving on to Chapter 2...

You read §2.1 on "Matrix Concepts. Other than matrix multiplication, all we'll need for now:

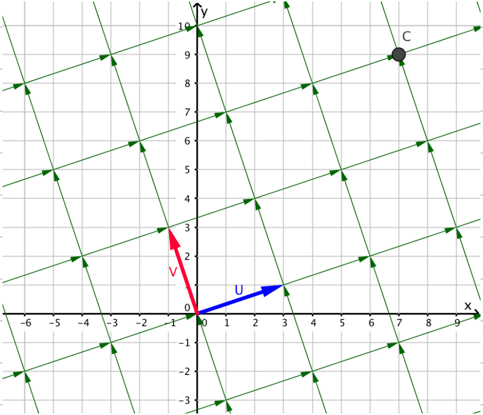
Lemma 2.1 Let $U, V \in \mathbb{R}^2$ be linearly independent. Then $\forall\, X \in \mathbb{R}^2$, $\exists$ unique $a, b \in \mathbb{R}$ such that

$$X = \underbrace{aU + bV}_{\text{linear comb'n}}$$

(see lemma for formulas for a, b. in particular...)

Lemma 2.4 Let $u, v \neq 0$ in $\mathbb{R}^2$, with $u \perp v$. Let $x \in \mathbb{R}^2$. Then

$$x = \frac{x \cdot u}{\|u\|^2} u + \frac{x \cdot v}{\|v\|^2} v$$



$c = 3u + 2v$

Corollary 2.5 (Same conditions) $\|x\|^2 = \frac{(x \cdot u)^2}{\|u\|^2} + \frac{(x \cdot v)^2}{\|v\|^2}$

Distances and Inequalities

We defined $\|x\| = \sqrt{x \cdot x} = \langle x, x \rangle^{1/2}$ $\Rightarrow$ $\left(\|x\|^2 = x \cdot x \right)$

Also,

$$\|Q-P\| = \underbrace{\text{dist from } P \text{ to } Q = \|P-Q\|}_{\forall u,\ \|u\| = \|-u\|} = |\overline{PQ}| = |\overline{QP}|$$

$\forall u,\ \|u\| = \|-u\|$ $\leftarrow$ warmup

<u>Def</u> $\overline{PQ} \cong \overline{RS}$ are <u>congruent</u> if $|\overline{PQ}| = |\overline{RS}|$.

<u>Prop</u> Congruence of line segments is equivalence relation.

Quick Aside: Equivalence Relations

Examples of Relations

$\mathbb{Z}, <$: $3<4$, $4 \nless 3$, $5 \nless 3$.

$\mathbb{R}, =$: $3=3$, $3 \neq 4$

$\mathbb{Z}, R$: aRb iff $a^2=b^2$

$(-3)R3$, 2 <u>not</u> rel'd to 3

A rel'n is an equivalence rel'n if it is...

1. reflexive: $\forall x,\ x \sim x$

2. symmetric: $\forall x,y$, if $x \sim y$ then $y \sim x$

3. transitive: $\forall x,y,z$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

You try: (I didn't type this up...)

Which of the following are equivalence relations?

people, "have same birthday" yes

lines, $\parallel$ yes

integers, $\leq$ not symm
$(1 \leq 2,\ 2 \nleq 1)$

lines, $\perp$ not refl., trans

$\mathbb{R}$, aRb if $a^2 = b^2$? yes

$\mathbb{R}$, $\approx$ probably not [not trans?]
appx equal to.

Cauchy-Schwarz (-Bunyakovsky) Inequality (Lemma 2.11)

$|u \cdot v| \leq \|u\| \cdot \|v\|$ with equality iff $u \parallel v$.

① Standard Pf

(done on board)

② Quicker Pf using lin. alg. concepts from Chapter 2.

(done on board)

Lemma 2.12 ($\triangle$ inequality):

(done on board)

(Restated in Prop 2.13 w/ line segments)

Thm 1.55 (Pythagorean) Let $A, B, C \in \mathbb{R}^2$ be distinct points.

(done on board)