Chapter 1 - Flythrough

This will be a fast review/intro — You're responsible for reading Chapter 1 (and 2) and talking to me if there's something you don't follow.

(Think: big ideas/terms, like "set," not the intricacies of, say, proof of Proposition 1.30...)

☆ I'll post a review of solving systems of linear eqns using subst'n, elimination or matrix multiplication.

Fundamentals / Vocabulary

$\forall$: for every, for all

$\exists$: there exists ($\exists!$: there exists a unique)

iff : iff and only if, $\Longleftrightarrow$

We won't use sets in much depth. Mostly:

$\mathbb{R}$ = real numbers $= \{x : x \in \mathbb{R}\}$

$\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$

and esp. subsets of those

Recall "Abstract" Function Notation

$f: A \to B$ — A: domain

$x \mapsto f(x)$ — B: codomain (image, range)

$x \rightsquigarrow f(x)$ — target (space)

"maps to"

Ex $f: \mathbb{R} \to \mathbb{R}$ or $\{x \in \mathbb{R} : x \geq 0\}$

$x \rightsquigarrow x^2$

Def **injective** or **one-to-one** (1:1): two elts in domain sent to different outputs.

surjective or **onto**: every elt of codomain is hit.

target

Functions	Injective	Surjective	Not a Function	
1.		✓		
2.	✓			
3.	✓			
4.	✓	✓		(bijection)
5.	✓			
6.	✓	$f(1) = (1,1) \neq (-1,1) = f(-1)$		
7.	✓			
8.				
9.				
10.			✓	

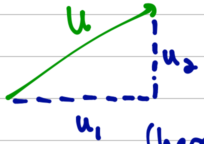
Vectors, Points and Lines

A (2D) vector is an ordered pair of real #'s, (a,b).

Common notations: $\langle a,b\rangle$, $\overrightarrow{(a,b)}$

Our book: $U = (u_1, u_2)$

Graphically, U is an arrow:



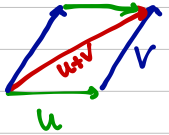
(here $u_1, u_2 > 0$)

Virtually every vector concept has an algebraic def^n/interpretation and a geometric/axiomatic one.

alg | geo

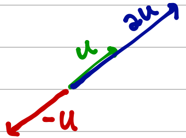
Addition

$U+V = (u_1, u_2) + (v_1, v_2)$
$\quad = (u_1+v_1,\ u_2+v_2)$



(scalar) mult'n

$cU = c(u_1, u_2)$
$\quad = (cu_1, cu_2)$



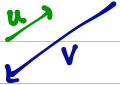
Subtraction

$U-V = U + (-1)V$
$\quad = (u_1-v_1,\ u_2-v_2)$

linearly dependent

$U = cV$ or $V = cU$

$\exists\ a, b$ such that
$aU + bV = 0\ (=(0,0))$
a, b not both 0.



(parallel)

alg | geo

dot product:
(inner product, scalar product)

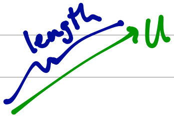
$$u \cdot v = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2 \neq (u_1 v_1, u_2 v_2)$$

?

$$u \cdot v = \langle u, v \rangle$$

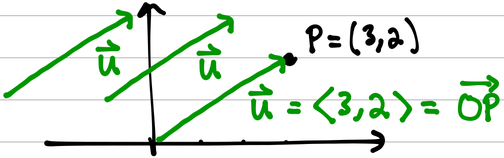
length/magnitude

$$\|u\| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2}$$



⚠ If you learned vectors from Stewart's book...

Stewart make huge distinctions between points and vectors:



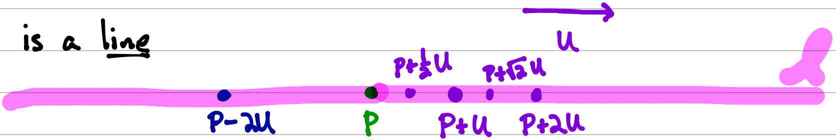
We won't. For us, vector and point are synonyms.

It's clear from context, and it makes life easier to do it this way. To wit:

Def Given a point P and non-zero vector u, the set

$$\ell = \{ P + su : s \in \mathbb{R} \}$$

is a line



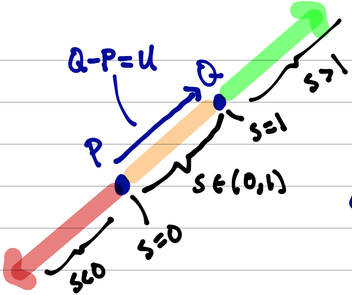
u is a direction indicator (dir vector). Points on ℓ are "incident" with line.

$P_1, P_2, \ldots, P_n$ are collinear if $\exists$ line incident w/ all of them.

Ex $(-1,2) + s(3,-4)$ $\quad (\frac{1}{2}, 0)$ is on line $(s = \frac{1}{2})$

$(5,6)$ is not. $\left.\begin{array}{l} -1+3s=5 \\ 2-4s=6 \end{array}\right\}$ no sol'n

$P + s(Q-P)$



$\overline{PQ} = \overline{QP}$

$\{P+s(Q-P)\} = \{Q+s(P-Q)\}$

$\overleftrightarrow{PQ} = \overleftrightarrow{QP}$

$\overrightarrow{PQ} \neq \overrightarrow{QP}$

$\|X\| = \sqrt{X \cdot X}$

length of $\overline{PQ}$ is $\|Q-P\| = \|P-Q\|$

$$= \sqrt{\langle Q-P, Q-P \rangle}$$

$$= \sqrt{(Q-P)\cdot(Q-P)}$$

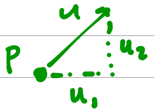
Does this def$^{\underline{n}}$ cover everything we expect?

- Can get segments, rays using restricted values of s.
- Two points form a line? Yes (wksheet)

Prop 1.4 two non-zero vectors are DI's of same line iff they're scalar mult's of each other.

Let $P \neq Q$. Then $\exists$ unique line $\overleftrightarrow{PQ}$ incident with both, $u = Q - P$ is a DI of $\overleftrightarrow{PQ}$, and every DI of $\overleftrightarrow{PQ}$ is difference of two pts on the line.

- slope of $P + sU$ is $\frac{u_2}{u_1}$, if $u_1 \neq 0$.



other forms: eliminate parameter

Ex $(-1,2)+s(3,-4) = (\underbrace{-1+3s}_{x(t)}, \underbrace{2-4s}_{y(t)})$

$x = 3s-1 \Rightarrow s = \frac{1}{3}(x+1)$

$y = -4s+2 \Rightarrow s = -\frac{1}{4}(y-2)$

$\frac{1}{3}(x+1) = -\frac{1}{4}(y-2)$

$4x+4 = -3y+6$ solve for y...

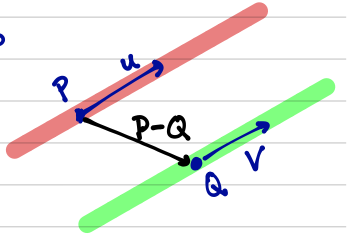
$3y = -4x+2$

$y = -\frac{4}{3}x + \frac{2}{3}$

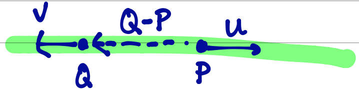
Def Two lines ℓ, m are <u>parallel</u>, $\ell \parallel m$ if DI's are $\parallel$.

Prop 1.6 Lines $\ell = \{P + sU\}$, $m = \{Q + tV\}$

- $\cap$ in <u>one</u> point if U, V lin. indep (not $\parallel$)
- empty $\cap$'n if $U \parallel V$ and $U \nparallel Q-P$



- are same line if $U \parallel V$ <u>and</u> $U \parallel Q-P$



Warmup Problem (9/13/17)

Recall: $U \cdot V = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2$

Prove: The dot product is commutative: $U \cdot V = V \cdot U$

$$U \cdot V = u_1 v_1 + u_2 v_2 = v_1 u_1 + v_2 u_2 = V \cdot U$$

Prove: The dot product is distributive: $U \cdot (V + W) = U \cdot V + U \cdot W$

$$\begin{aligned}
U \cdot (V + W) &= (u_1, u_2) \cdot (v_1 + w_1, v_2 + w_2) \\
&= u_1(v_1 + w_1) + u_2(v_2 + w_2) \\
&= u_1 v_1 + u_1 w_1 + u_2 v_2 + u_2 w_2 \\
&= (u_1 v_1 + u_2 v_2) + (u_1 w_1 + u_2 w_2) \\
&= U \cdot V + U \cdot W
\end{aligned}$$

Quick Status Check: which of Euclid's Axioms work so far?

1. Given two pts, $\exists$ line containing them ✓
2. Lines can be extended indefinitely ✓
3. Given A, B, $\exists$ circle cent'd at A with radius $\overline{AB}$. ✓

   $r = \|B - A\|$

   $C = \{X : \|X - A\| = r\}$

4. Right angles are all equal ✗
5. $\parallel$ postulate (✓ HW)

Perependicularity / Orthogonality

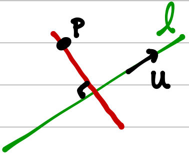
Def $U \perp V$ if $U \cdot V = u_1 v_1 + u_2 v_2 = 0$. Two lines ℓ, m are perpendicular if they have $\perp$ Direction Indicators.

$\parallel$ and $\perp$ play important roles...

Corollary 1.11 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\parallel$ to ℓ.

Prop 1.15 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\perp$ to ℓ.

Prop 1.15 If ℓ is a line and P is a point, $\exists$ exactly one line incident with P and $\perp$ to ℓ.

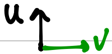


Pf Suppose $\ell: Q+sU$, $U \neq 0$

Claim: $U=(u_1,u_2) \Rightarrow V=(-u_2,u_1) \perp U$

now show any vector $\perp U$ is scalar mult. of this.

Case 1 $U=(u_1,u_2)$, $u_1=0 \Rightarrow V=(-u_2,0)$



$U \cdot V = (0,u_2)\cdot(-u_2,0) = 0+0=0$ ✓

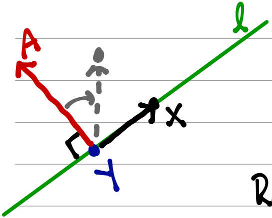
Case 2 $u_2=0$

Case 3 in gen'l, $U \cdot V = (u_1,u_2)\cdot(-u_2,u_1) = -u_1u_2+u_1u_2=0$

$m: P+sV$ incident w/ P, $\perp \ell$.

Normal Form

Given line ℓ, choose $Y \in \ell$ and $A \perp \ell$ (i.e. $A \perp U$, U any DI of ℓ). Then

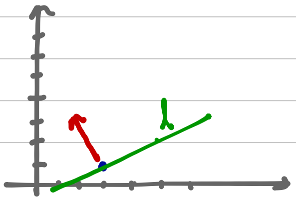


$$\ell = \{X : \underbrace{A \cdot (X - Y) = 0}_{\text{nl eqn of line}}\}$$

Remember A, Y fixed; $X = (x_1, x_2)$ $\left(= (x, y)\right)$ is variable. If $\|A\| = 1$, this is "special" nl eqn.

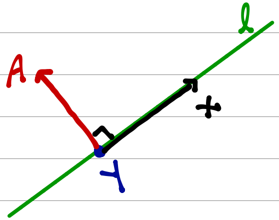
Ex $\quad Y = (3,1)$, $A = (-1,2)$

$$(-1,2) \cdot (X - Y) = (-1,2) \cdot \left((x_1, x_2) - (3,1) \right)$$
$$= (-1,2) \cdot (x_1 - 3, x_2 - 1)$$
$$= -x_1 + 3 + 2x_2 - 2 = 0$$



$$2x_2 = x_1 - 1$$
$$x_2 = \tfrac{1}{2}x_1 - \tfrac{1}{2} \qquad y = \tfrac{1}{2}x - \tfrac{1}{2}$$

$\exists$ alternate version of normal form:



$$A\cdot(X-Y)=0$$

$$A\cdot X - A\cdot Y = 0$$

$$A\cdot(x,y) - c = 0$$

$$A\cdot X = c \qquad \left[A\cdot(x_1,x_2)=c\right]$$

Ex $A=(-1,2)$, $Y=(3,1)$

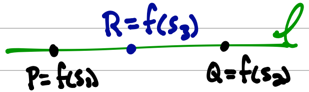
$$A\cdot(X-Y)=0$$

$$A\cdot X - \underbrace{A\cdot Y}_{=-1} = 0 \Rightarrow A\cdot X + 1 = 0$$

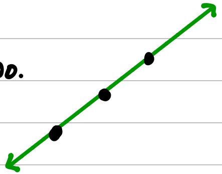
$$A\cdot X = -1$$

Betweeness

Def/Prop Let $f(s)$ be eqn of line, $f(s_1)=P$, $f(s_2)=Q$. Then R is between P, Q if $\exists\ s_3$, $s_1 < s_3 < s_2$, $f(s_3) = R$.

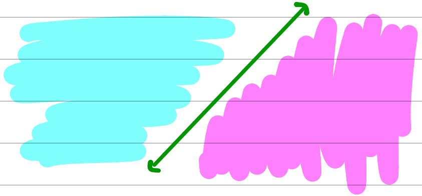


Corollary 1.22 Given 3 pts on a line, one must be b/w other two.

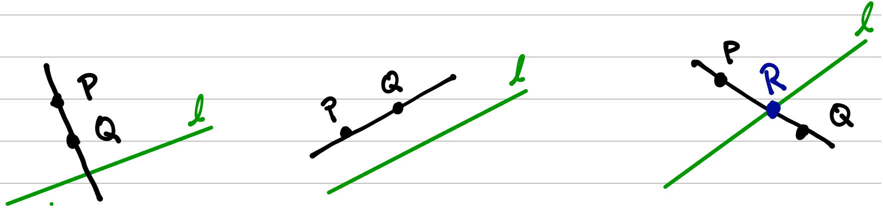


See book for further corollaries with 4+ points

A line separates $\mathbb{R}^2$ into two "half planes."



<u>Clever Def</u> $P, Q \notin \ell$ on opposite sides of ℓ if $\exists\ R \in \ell$ between them. Otherwise, they're on the same side.

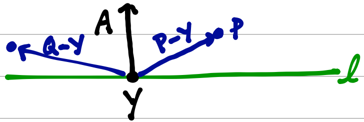


Prop 1.30 Let $\ell: A\cdot(X-Y)=0$ $(Y\in\ell,\ A\perp\ell)$ and $P, Q\notin\ell$. Then P and Q are on same/opposite side of ℓ if $A\cdot(P-Y)$, $A\cdot(Q-Y)$ have same signs.

⚠ Book uses $A\cdot X=c$, compares $A\cdot P-c$, $A\cdot Q-c$.

⚠!! Would be "simple" (well, simpler) if we had

$$\vec{a}\cdot\vec{b}=\|\vec{a}\|\cdot\|\vec{b}\|\cos\theta \quad \begin{array}{l} >0 \text{ for } \theta\in[0,\pi/2) \\ <0 \text{ for } \theta\in(\pi/2,\pi] \end{array}$$



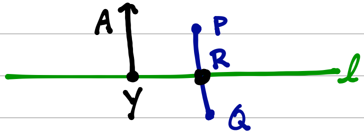
Prop 1.30 Let $\ell: A\cdot(X-Y)=0$ $(Y\in\ell,\ A\perp\ell)$ and $P, Q\in\ell$. Then P and Q are on same/opposite side of ℓ if $A\cdot(P-Y)$, $A\cdot(Q-Y)$ have same/opposite sign.

Pf Let $g(s) = A\cdot\left(\underbrace{[P+s(Q-P)]}_{\text{line segment } \overline{PQ}} - Y\right)$ for $0\le s\le 1$

$g(s)=0$ for some s iff $\overline{PQ}$ intersects ℓ at pt R
($\Rightarrow P, Q$ opposite sides)

Now $g(s) = \underbrace{A\cdot(P-Y)}_{\#} + s\,\underbrace{A\cdot(Q-P)}_{\#} = b + ms = ms+b$

That's linear, so min/max values @ endpts, $g(s)=0$ only if $g(0)$, $g(1)$ have diff. signs.



$g(0) = A\cdot(P-Y)$, $g(1) = A\cdot(Q-Y)$.

Second Warmup Question:

Prove: $(cU)\cdot V = c(U\cdot V)$

<u>Pf</u>: $(cU)\cdot V = (cu_1, cu_2)\cdot(v_1, v_2)$

$$= cu_1v_1 + cu_2v_2$$

$$= c(u_1v_1 + u_2v_2)$$

$$= c(U\cdot V)$$

Moving on to Chapter 2...

You read §2.1 on "Matrix Concepts. Other than matrix multiplication, all we'll need for now:

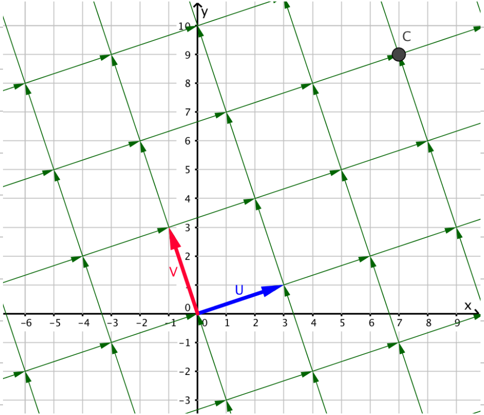
Lemma 2.1 Let $U, V \in \mathbb{R}^2$ be linearly independent. Then $\forall\, X \in \mathbb{R}^2$, $\exists$ unique $a, b \in \mathbb{R}$ such that

$$X = \underbrace{aU + bV}_{\text{linear combination of } U, V.} \qquad \text{"coords } (a,b)\text{"}$$

(See lemma for formulas for a,b. in particular...)

Lemma 2.4 Let $u, v \neq 0$ in $\mathbb{R}^2$, with $u \perp v$. Let $x \in \mathbb{R}^2$. Then

$$x = \frac{x \cdot u}{\|u\|^2} u + \frac{x \cdot v}{\|v\|^2} v$$



Corollary 2.5 (Same conditions) $\|x\|^2 = \frac{(x \cdot u)^2}{\|u\|^2} + \frac{(x \cdot v)^2}{\|v\|^2}$

Distances and Inequalities

We defined $\|x\| = \sqrt{x \cdot x} = \langle x, x \rangle^{1/2}$

Also,

$$\underbrace{\|Q-P\| = \text{dist from } P \text{ to } Q = \|P-Q\|}_{\forall u,\ \|u\| = \|-u\|} = |\overline{PQ}| = |\overline{QP}|$$

<u>Def</u> $\overline{PQ} \simeq \overline{RS}$ are <u>congruent</u> if $|\overline{PQ}| = |\overline{RS}|$.

<u>Def</u> Congruence of line segments is equivalence relation.

Quick Aside: Equivalence Relations

Examples of Relations

$\mathbb{Z}$, $<$: $3<4$, $5 \nless 1$

$\mathbb{R}$, $=$: $3=3$, $3\neq 4$

$\mathbb{R}$, aRb if $a^2=b^2$

A rel'n is an equivalence rel'n if it is...

1. reflexive: $\forall x,\ xRx$
2. symmetric: $\forall x,y$, if xRy then yRx.
3. transitive: $\forall x,y,z$, if xRy and yRz, then xRz

You try: (I didn't type this up...)

Which of the following are equivalence relations?

people, "have same birthday" yes

lines, $\parallel$ yes

integers, $\leq$ ✗ not symm

lines, $\perp$ ✗ not reflexive or trans.

$\mathbb{R}$, aRb if $a^2=b^2$ yes

$\mathbb{R}$, $\approx$ (?)

appx equal to.

Cauchy-Schwarz (-Bunyakovsky) Inequality (Lemma 2.11)

$|u \cdot v| \leq \|u\| \cdot \|v\|$ with equality iff $u \parallel v$.

① Standard Pf

② Quicker Pf using lin. alg. concepts from Chapter 2.

(These were done on board)

end day 2.

Lemma 2.12 (Δ inequality):

(done on board)

(Restated in Prop 2.13 w/ line segments)

Thm 1.55 (Pythagorean) Let $A, B, C \in \mathbb{R}^2$ be distinct points.

(done on board)