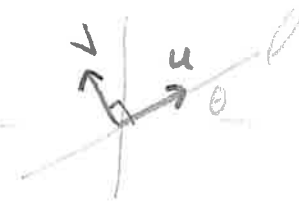
Math 5335 — Classification of Isometries — F'19

This is all in the book, but presented in a different order or with a different viewpoint/approach (e.g. the formula for reflections). So follow class notes instead.

We know from Chapter 4: $\mathcal{U}:\mathbb{R}^2 \to \mathbb{R}^2$ is an isometry

iff $\mathcal{U}(X) = MX + P$, $M \in \{R_\theta, F_\theta\}$

From "Useful Facts": Let $U = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, $V = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$



Then $F_\theta U = U$, $F_\theta V = -V$

(and $F_\theta(kU) = kU$, $F_\theta(kV=) = -kV$)

Overriding Questions: How many isometries are there? What are they? How do we know that list is complete?

Let's start with a few examples

<u>Identity</u> $\mathcal{U}(X) = X \quad (= R_0 X + 0)$

<u>Translation</u> $\mathcal{T}_V(X) = X + V \quad (= I X + V = R_0 X + V)$

(We saw those $\mathcal{T}$ in Chapter 4.)

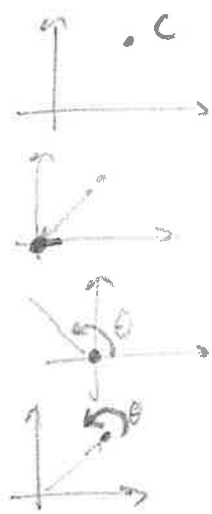
<u>Rotation by θ about C</u> Seems hard in gen'l - so reduce to a previously solved problem! By construction, R_θ rotates $\mathbb{R}^2$ by θ about the origin.

<u>Let's do this in steps!</u>

(1) Move C to origin: $X - C$

(2) Rotate by θ about O: $R_\theta(X-C)$

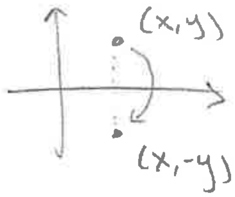
(3) Move O back to C: $R_\theta(X-C)+C$



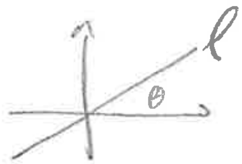
(Recall, from HW - composition of isometries is an isometry)

$$\boxed{\mathcal{R}_{\theta,C}(X) = R_\theta(X-C)+C}$$

Reflections: Also tricky. Let's do stages: refl'n across x-axis, across a line through O, across any line.

x-axis 

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}ax+by\\ cx+dy\end{bmatrix}\begin{bmatrix}x\\ -y\end{bmatrix} \Rightarrow \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

ℓ through O 

① Rotate by $-\theta$ $\quad R_{-\theta}X$ 

② Reflect across x-axis. $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}R_{-\theta}X$

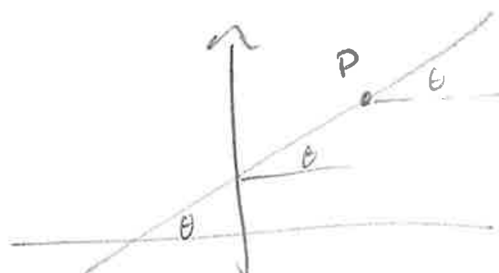
③ Rotate back : $R_\theta\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}R_{-\theta}X$ 

Note $R_\theta\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}R_{-\theta}\right) = \begin{bmatrix}c\theta & -s\theta\\ s\theta & c\theta\end{bmatrix}\begin{bmatrix}c\theta & s\theta\\ s\theta & c\theta\end{bmatrix} = \begin{bmatrix}c^2\theta - s^2\theta & 2s\theta c\theta\\ 2c\theta s\theta & s^2\theta - c^2\theta\end{bmatrix} = \begin{bmatrix}c2\theta & s2\theta\\ s2\theta & -c2\theta\end{bmatrix} = F_\theta$.

Thus $T(X) = F_\theta X$ reflects $\mathbb{R}^2$ across line which forms angle of θ w/ x-axis.

General Refl'n across ℓ

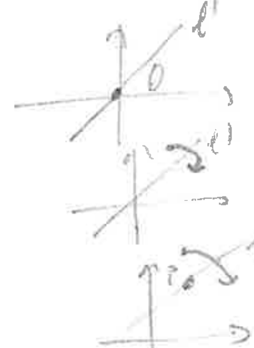
(which contains a pt P, forms angle of θ w/ horizontal)



① Move P to origin : $(X-P)$

② Reflect : $F_\theta(X-P)$

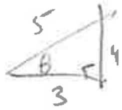
③ Move O back to P : $F_\theta(X-P)+P$



<u>Ex</u> Get a pt $\in$ Quad II, $u = (3,4)$

$c2\theta = c^2\theta - s^2\theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$

$s2\theta = 2c\theta s\theta = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$

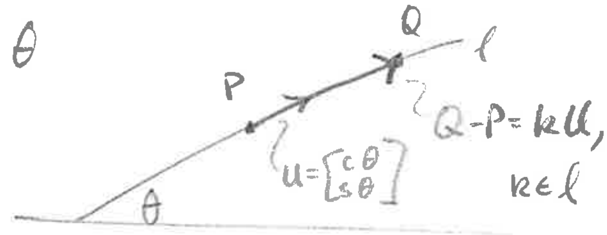


$$M_\ell(X) = \begin{bmatrix} -7/25 & 24/25 \\ 24/25 & +7/25 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \; \end{bmatrix}\right) + \begin{bmatrix} \; \end{bmatrix}$$

⚠ Our formula for M_ℓ seems to depend on arbitrary choice of $P \in \ell$. Uh-oh (<u>ex</u> with other pt above on ℓ).

<u>Prop</u> Let $P, Q \in \ell$, which forms angle of θ w/ the horizontal. Then

$$F_\theta(X-P)+P = F_\theta(X-Q) = Q$$



(And thus we can use <u>any</u> pt on ℓ in formula for M_ℓ.)

<u>Pf</u> By "Useful Facts," $F_\theta(Q-P) = Q-P$

<u>Method 1</u>

$$\left.\begin{aligned} F_\theta(X-P)+P &= F_\theta X - F_\theta P + P \\ F_\theta(X-Q)+Q &= F_\theta X - F_\theta Q + Q \end{aligned}\right\}$$

equal if $-F_\theta P + P = -F_\theta Q + Q$

$F_\theta(Q-P) = Q-P$ ✓

<u>Method 2</u>

$$F_\theta(Q-P) = Q-P$$
$$F_\theta(X-X+Q-P) = Q-P$$
$$F_\theta(X-P) - F_\theta(X-Q) = Q-P$$
$$F_\theta(X-P)+P = F_\theta(X-Q)+Q$$

⚠ Two sided pfs

<u>Method 3</u>

$$\begin{aligned} F_\theta(X-P)+P &= F_\theta(X-Q+Q-P) \\ &= F_\theta(X-Q) + F_\theta(Q-P) + P \\ &= F_\theta(X-Q) + Q-P+P \\ &= F_\theta(X-Q)+Q \end{aligned}$$

Combining / Composing Isometries

$\boxed{\text{Kaleidescope/Hubcap Activity}}$ — Symmetry Groups (separate slides posted)

<u>Prop</u> $R_{\varphi,c} \circ R_{\theta,c} = R_{\varphi+\theta,c}(x)$

<u>Pf</u> $R_\varphi\left(\left[R_\theta(x-c)+c\right]-c\right)+c = R_\varphi R_\theta(x-c)+c$

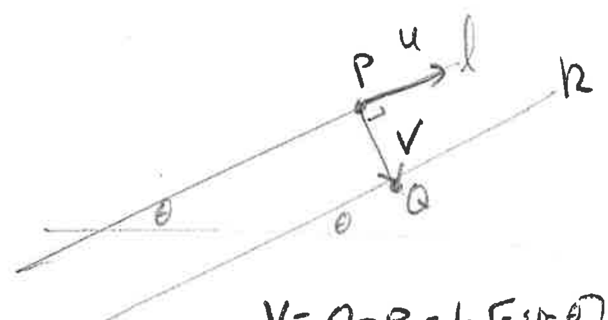
$$= \underbrace{R_{\varphi+\theta}}_{\text{Ch 5}}(x-s)+c$$

<u>Prop</u> $m_\ell \circ m_\ell = id$ (<u>Remark</u>: if $U \circ U = id$, U is an <u>involution</u>)

<u>Pf</u> $F_\theta\left(\left[F_\theta(x-P)+P\right]-P\right)+P = F_\theta F_\theta(x-P)+P$ ($F_\theta F_\theta = I$ - useful facts)

$$= x-P+P$$

$$= x$$



<u>Prop</u> Let $\ell \parallel k$ as shown. Then

$$m_k \circ m_\ell(x) = T_{2v}(x).$$

$$v = Q - P = k\begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$$

<u>Pf</u> $m_k(m_\ell(x)) = m_k\left(F_\theta(x-P)+P\right)$

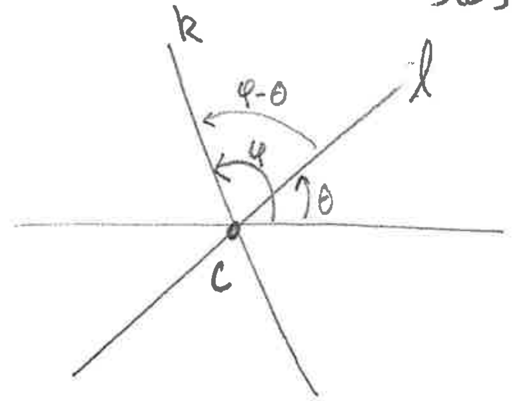
$$= F_\theta\left[\left(F_\theta(x-P)+P\right)-Q\right]+Q$$

$$= F_\theta F_\theta(x-P) - F_\theta(Q-P) + Q$$

$$= (x-P)+(Q-P)+Q$$

$$= x + 2(Q-P)$$

$$= x + 2v$$



<u>Prop</u> If $l \cap k \{c\}$, then

$$M_k \circ M_l = R_{2(\varphi-\theta),c}$$

where l, k form angles of θ, φ with the horizontal.

<u>Remark</u> Alternatively, the rotation is by <u>twice</u> the angle from l to k. (in a counterclockwise direction - we now have positive, negative angles after Chapter 5!)

<u>Pf</u>

$$\begin{aligned} M_k \circ M_l(X) &= M_k\left(F_\theta(X-c)+c\right) \\ &= F_\varphi\left(\left[F_\theta(X-c)\cancel{+c}\right]\cancel{-c}\right)+c \\ &= \underbrace{F_\varphi F_\theta}_{=R_{2(\varphi-\theta)}\ (\text{"useful facts"})}(X-c)+c \\ &= R_{2(\varphi-\theta)}(X-c)+c \\ &= R_{2(\varphi-\theta),c}(X). \end{aligned}$$

<u>Back to rotations</u> What about $\mathcal{R}_{\varphi,D} \circ \mathcal{R}_{\theta,C}$ with $C \neq D$?

<u>Prop</u> $\mathcal{R}_{\pi,D} \circ \mathcal{R}_{\pi,C} = \mathcal{T}_{2(D-C)}$. (GeoGebra Demo.)

<u>Pf</u> Note $R_\pi = \begin{bmatrix} c\pi & -s\pi \\ s\pi & c\pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $R_\pi X = R_\pi \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$

$$\begin{aligned}
\mathcal{R}_{\pi,D}\left(\mathcal{R}_{\pi,C}(X)\right) &= \mathcal{R}_\pi\left(R_\pi(X-C)+C\right) \\
&= \mathcal{R}_\pi\left(C - X + C\right) \\
&= R_\pi\left([2C-X]-D\right)+D \\
&= X - 2C + D + D \\
&= X + 2(D-C).
\end{aligned}$$

General case (where angles aren't both π) is harder!

(Turn for the worse activity).

$\equiv$ D2

Ok - rather than test every possible combination, we need a systematic approach. The <u>key</u> is:

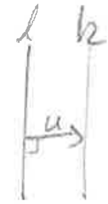
★ <u>Thm 6.16</u> Every isometry can be expressed (constructed) as the composition of ≤ 3 refl'ns.

"Pf": Lab 3.

Remark: Thus we "just" need to figure out all possibilities for $n=1,2,3$ refl'ns.

$\underline{n=1}$ M_ℓ refl'n

$\underline{n=2}$ $M_k \circ M_\ell$ is either: • T_{2u} if $\ell \parallel k$



• $R_{2\theta,c}$ if $\ell \cap k = \{c\}$

Special cases: $\ell = \ell$: $M_\ell \circ M_\ell = Id(X) = T_0(X) = R_{0,c}(X)$

$\ell \perp k$: $M_k \circ M_\ell = E_c(X)$ 'central inversion'

$(= R_{\pi,c}(X))$

$\underline{n=3}$ $M_k \circ M_\ell \circ M_m = \dots ?$

$\exists$ new possibility!

Def Given $u \parallel \ell$, $G(X) = g(X) = M_\ell \circ T_u(X)$ is a glide refl'n.

Ex

Prop Given $u \parallel \ell$, $M_\ell \circ T_u = T_u \circ M_\ell$

(So we can do transl'n/refl'n in a glide refl'n in either order)

Pf $M_\ell \circ T_u(X) = F_\theta([X+u]-P)+P$

$$= F_\theta X + \underbrace{F_\theta u}_{=u} - F_\theta P + P$$

($F_\theta u = u$ - "Useful Facts")

$$= (F_\theta(X-P)+P) + u$$

$$= T_u(M_\ell(X))$$

(Glide Refl'ns Sheet)

⚠ How do we know $g(x)$ isn't a refl'n in hiding?
Key turns out to be <u>fixed pts</u>

Def A <u>fixed pt</u> of a fn $f: A \to A$ is a pt $a \in A$ s.t. $f(a) = a$.

Ex $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 6-2x$. If $f(x) = x$, then $6-2x = x \Rightarrow x = 2$

<u>What about transl'ns, rot'ns and refl'ns?</u>

Prop If $u \neq 0$, T_u has <u>no</u> fixed pts.

Pf if $X+u = X$, $u = X - X = 0$

$(\theta \neq 2\pi n)$

Prop Only fixed pt of $R_{\theta,0}(X)$ is $X=0$.

Pf $R_{\theta,0}(X) = RX = X \Rightarrow R_\theta X - X = 0$

$$\underbrace{(R_\theta - I)}_{A \text{ - is invertible (check)}} X = 0$$

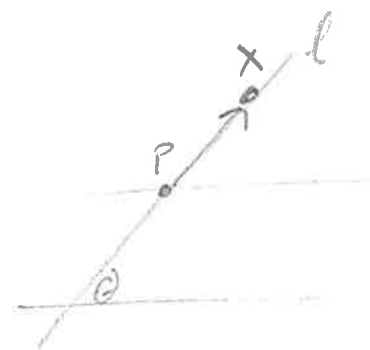
$\Rightarrow X = A^{-1}0 = 0.$

Prop Only fixed pt of $R_{\theta,C}(X)$ is $X=C$ $\quad(\theta \neq 2\pi n)$

Pf $R_{\theta,C}(X) = X \Rightarrow R_\theta(X-C)+C = X$

Then $R_\theta(X-C) = (X-C)$

$\Rightarrow X - C = 0$ so $X = C$.



Prop Fixed pts of $M_\ell(X)$ are pts on ℓ

Pf If $X = F_\theta(X-P)+P$, then

$F_\theta(X-P) = (X-P)$

We've seen (you check) fixed pts/vectors of F_θ are those $\parallel \ell$, so $X-P$ is DV of ℓ and

$X = P + (X-P) \in \ell$.

<u>Prop</u> A glide refl'n $g(X)$ has <u>no</u> fixed pts.

<u>Pf</u> $g(X) = T_u(M_\ell(X)) = F_\theta(X-P)+P+u.$

Suppose $F_\theta(X-P)+P+u = X$, so

$$F_\theta(X-P) = (X-P)+u. \qquad (1)$$

Mult. by F_θ $\quad F_\theta F_\theta(X-P) = F_\theta(X-P)+u \qquad (F_\theta u = u)$

$$(X-P)-u = F_\theta(X-P) \qquad (2)$$

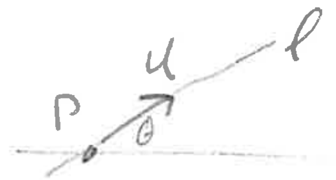
So, if $g(X)$ has fixed pt, (1) and (2) both true:

$$F_\theta(X-P) = (X-P)+u = (X-P)-u$$

i.e. $u = -u \Rightarrow u = 0 \Rightarrow g$ a <u>refl'n</u>.

(or if $u=0$, $g(X)$ <u>not</u> a glide refl'n)

<u>Prop</u> $\mathcal{G}(X)$ is not a transl'n, rot'n or refl'n. It's a "new" kind of isometry.



<u>Pf</u> $\mathcal{G}(X) = \mathcal{T}_u(\mathcal{M}_\ell(X)) = F_\theta(X-P)+P+u$

$= F_\theta X + (\text{stuff})$

Thus the matrix for $\mathcal{G}(X)$ is F_θ, so it's <u>not</u> a transl'n or rot'n.

By above, $\mathcal{G}(X)$ has no fixed pts, so <u>not</u> a refl'n.

<u>Great</u> - we now have 4 isom's: transl'ns, rot'ns, refl'ns and glide refl'ns. Are there any others I can get with 3 refl'ns?

Consider $\mathcal{M}_k \circ \mathcal{M}_\ell \circ \mathcal{M}_m$

either of these compositions of 2 refl'ns can be rewritten as a transl'n or rot'n - or maybe even $\mathrm{id}(X) = \mathcal{T}_0(X) = \mathcal{R}_{0,c}(X)$.

So we have to consider 4 possibilities:

(1) transl'n ∘ refl'n
(2) refl'n ∘ transl'n

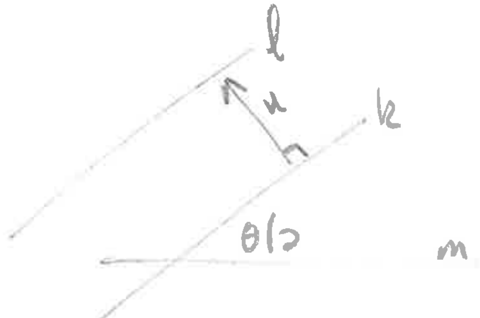
already know are GR's *if* transl'n ∥ mirror. We'll have to consider other cases.

(3) rot'n ∘ refl'n
(4) refl'n ∘ rot'n.

we can rewrite as compositions of transl'n and refl'n (!!), so we don't have to worry about (3), (4) !!

<u>Ex</u> Consider (4), $\mathcal{M}_\ell \circ \mathcal{R}_{\theta,C}$. $(= \mathcal{M}_\ell \circ \mathcal{M}_k \circ \mathcal{M}_m)$

$\mathcal{R}_{\theta,C}$ can be constructed by reflecting across <u>any</u> two lines intersecting at C, forming angle of $\theta/2$ from 1st line to 2nd. Let's choose so that 2nd line is $\parallel \ell$:



$$\mathcal{M}_\ell \circ \mathcal{R}_{\theta,C} = \mathcal{M}_\ell \circ (\mathcal{M}_k \circ \mathcal{M}_m)$$
$$= (\mathcal{M}_\ell \circ \mathcal{M}_k) \circ \mathcal{M}_m$$
$$= \mathcal{T}_{2u} \circ \mathcal{M}_m$$

which is case (1) above.

Similarly, (3) can be rewritten as (2).

Thus, we only need to check $\mathcal{T}_u \circ M_\ell$ and $M_\ell \circ \mathcal{T}_u$ to see if we get any new isometries

Triple Refl'ns Sheet

1st Case $u \perp \ell$

Prop Let $u \perp \ell$. Then $\mathcal{T}_u \circ M_\ell = M_k$, where $k = \mathcal{T}_{u/2}(\ell)$
(i.e. k is the line ℓ, translated by $u/2$.)

Pf Let $P \in \ell$.

⚠ We know $F_\theta u = -u$
or $u = -F_\theta u$
("Useful Facts")

$$
\begin{aligned}
\mathcal{T}_u \circ M_\ell(X) &= \mathcal{T}(F_\theta(X-P)+P) \\
&= F_\theta(X-P)+P+u \\
&= F_\theta(X-P)+P+\tfrac{1}{2}u+\tfrac{1}{2}u \\
&= F_\theta(X-P)+P-\tfrac{1}{2}F_\theta u+\tfrac{1}{2}u \\
&= F_\theta\left(X-P-\tfrac{1}{2}u\right)+P+\tfrac{1}{2}u \\
&= F_\theta(X-Q)+Q, \quad Q = P+\tfrac{1}{2}u \\
&= M_k(X), \text{ for } k = \mathcal{T}_{u/2}(\ell).
\end{aligned}
$$

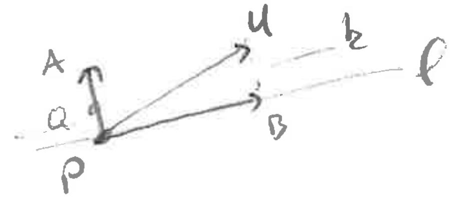
You check: $M_\ell \circ \mathcal{T}_u$ (other order) also a refl'n.

2nd Case U not $\parallel$ or $\perp$ ℓ (General Case)

Write U as $U = A + B$, where $A \perp \ell$, $B \parallel \ell$.

Let $P \in \ell$, so $M_\ell(X) = F_\theta(X-P)+P$.

We know $F_\theta A = -A$, $F_\theta B = B$.



Let's check $T_U \circ M_\ell(X)$, which is

$$F_\theta(X-P)+P+U = F_\theta(X-P)+P+\underbrace{\tfrac{1}{2}A+\tfrac{1}{2}A+B}_{U}$$

$$= F_\theta(X-P)+P+\tfrac{1}{2}A-\tfrac{1}{2}F_\theta A+F_\theta B$$

$$= F_\theta\left(X-P-\tfrac{1}{2}A+B\right)+P+\tfrac{1}{2}A \qquad Q = P+\tfrac{1}{2}A.$$

$$= F_\theta\left([X+B]-Q\right)+Q$$

$$= M_k\left(T_B(X)\right) = G(X) \quad \text{mirror } k = T_{A/2}(\ell) \text{ glide } B.$$

<u>We've proven:</u>

<u>Prop</u> With above setup, $T_U \circ M_\ell = G(X)$ with

<u>You check:</u> $M_\ell \circ T_U$ is also glide refl'n.

We have (finally!) exhausted all possibilities, and have proven:

<u>Thm</u> The only possible isometries of $\mathbb{R}^2$ are:

- $\mathcal{I}(X)$, the identity.

  ort'n preserving, involution, fixed pts $= \mathbb{R}^2$.
  comp'n of 0 or 2 refl'ns ($\mathcal{I} = \mathcal{M}_\ell \circ \mathcal{M}_\ell \ \forall \ell$)

- $\mathcal{M}_\ell(X)$, refl'n across ℓ

  ort'n <u>reversing</u>, involution, fixed pts $= \ell$.

- $\mathcal{R}(X)$ rot'n by θ about C

  ort'n preserving, not invol'n (unless $\theta = 0, \pi$), fixed pts $= \{C\}$ (unless $\theta = 0$). Comp'n of 2 refl'n.

  <u>special</u> rot'ns: $\mathcal{R}_{0,C}(X) = \mathcal{I}(X)$, $\mathcal{R}_{\pi,C}(X) = \mathcal{E}_C(X) = 2C - X$ (invol'n)

  rot'ns by $\theta \notin \{0, \pi\}$ called <u>non-special</u>.

- $\mathcal{T}_U(X)$, transl'n by U

  ort'n preserving, non invol'n, <u>no</u> fixed pts unless $U = 0$, which is <u>degenerate</u>: $\mathcal{T}_0(U) = \mathcal{I}(X)$. Comp'n of two refl'ns.

- $\mathcal{G}(X)$, glide refl'n, glide by U, refl'n across ℓ, $U \parallel \ell$.

  ort'n <u>reversing</u>, not invol'n, <u>no</u> fixed pts (<u>unless $U=0$</u>)
  comp'n of <u>three</u> refl'ns.

  degenerate glide refl'n