Chapter 6 - Classification of Isometries

This material is all in the book, but presented in a different order or with a different viewpoint/approach. (e.g. the formula for reflections) Take good notes, and follow class notes instead of book.

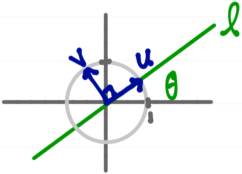
Overriding Questions: How many isometries are there? What are they? How do we know that list is complete?

We know (Ch 4) $U:\mathbb{R}^2 \to \mathbb{R}^2$ isometry iff $U(X) = MX + P$ where M is

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad F_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

We know ("Useful Facts") $R_\varphi R_\theta = R_{\varphi+\theta}$, $F_\varphi F_\theta = R_{2(\varphi-\theta)}$

Let $U = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $V = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$



Then: $F_\theta U = U$ and $F_\theta V = -V$

and $F_\theta(kU) = kU$, $F_\theta(kV) = -kV$ $\forall k \in \mathbb{R}$

Trig identities: $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

if $a=b=\theta$: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

Our Path through Chapter 6

Formulas for basic isometries: transl'ns, rot'ns, refl'ns

Composition of isometries:

Symmetry Groups. Basic combinations of rot'ns, refl'ns.
Examples of more complicated combinations

Systematic Approach via reflections: Glide Reflections

Fixed Points: Proving glide refl'ns are new, and our list is complete.

Conclusion: $G = \{R_0, R_{120}, R_{240}, L_1, L_2, L_3\}$, $*$ is a **group**:

Def$^{\underline{n}}$ A group is a set G with an operation $\bullet$ satisfying:

closure: $\forall\, a, b \in G,\quad a \bullet b \in G$

identity: $\exists\, e \in G$ s.t. $\forall\, a \in G,\quad a \cdot e = e \cdot a = a$

inverses: $\forall\, a \in G,\quad \exists\, b \in G$ s.t. $a \bullet b = b \bullet a = e$

associativity: $\forall\, a, b, c \in G,\quad (a \bullet b) \bullet c = a \bullet (b \bullet c)$

Ex The above group is known as D_3 (or D_6), the symmetry group of the equilateral triangle.

Def A symmetry of a set $S \subseteq \mathbb{R}^2$ is an isometry $\mathcal{U}$ s.t.

$$\forall x \in S, \ \mathcal{U}(x) \in S. \qquad \left(\text{So } \mathcal{U}(S) = S\right)$$

The set of all symmetries of S is called the symmetry group of S.

Question: Who Cares?

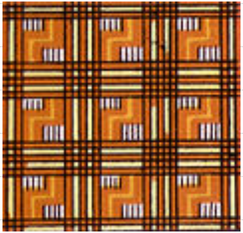
Ex An eq. △ and a square are fundamentally different...

... because their symmetries are different (R_{120} vs R_{90})

Big whoop. We need groups and isometries for that?

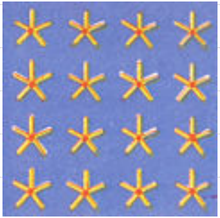
Ex Ok. What about these (infinite) patterns?

I



II

III



IV

I, II, III are "same"!
(same symm. group, p2)

IV is different (pm)

An eq. $\triangle$ has a finite symmetry group. ($|D_6| = 6$)

Wallpaper patterns have infinite symmetry groups.

Thm $\exists$ exactly 17 kinds of wallpaper.

(17 "wallpaper groups")

☆ For us: Composing isometries results in isometrys
R's, F's – reflections?
F's, F's – rot'ns?