Math 5335 Chapter 6 – Classification of Isometries (F17)

Our goal: find all possible isometries $\mathcal{U}: \mathbb{R}^2 \to \mathbb{R}^2$ — and show that our list is complete!

We know (Chapter 4): $\mathcal{U}(X) = MX + P$, $M \in \{R_\theta, F_\theta\}$

where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $F_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

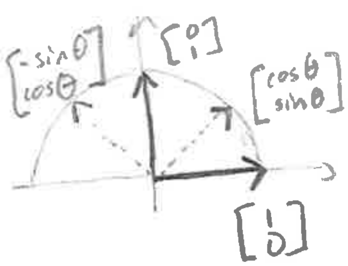
Let's start with a few basic examples... (some we've seen in ch 4...)

I Identity: $\mathcal{I}(X) = X$ $\left(= IX + 0, \text{ where } I = R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

II Translation by V: $\mathcal{T}_V(X) = X + V$ $\left(= IX + V \right)$

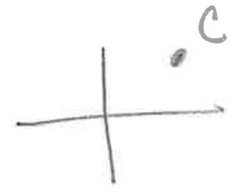
III Rotation by θ centered at C:

We know how to rotate $\mathbb{R}^2$ about the origin:



The matrix R_θ, by design, sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$. In other words, it rotates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by θ. Hence (using linear alg) it rotates all of $\mathbb{R}^2$.

So to rotate about C, we reduce to the previously solved problem of rotating about O.
Let's build it up, step by step.



Step 0 (beginning)

Step 1 Move C to O: $\mathcal{U}(X) = X - C$

Step 2 Rotate about O: $\mathcal{U}(X) = R_\theta(X-C)$

Step 3 Move back to C: $R_\theta(X-C)+C$

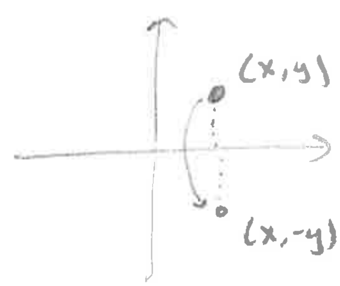
overall effect is to rotate by θ about C:

$$\mathcal{R}_{\theta,C}(X) = R_\theta(X-C)+C \quad \Big(= \underbrace{R_\theta X + (-R_\theta C + C)}_{\text{less useful form.}} \Big)$$

IV Reflection across line ("mirror") ℓ

That seems hard - but again, we can do a simple case.

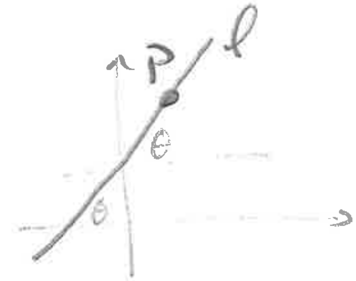
$$F_0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



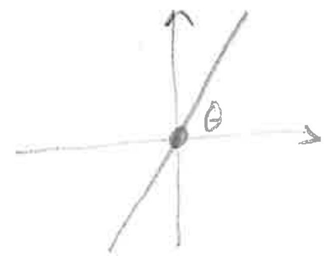
So F_0 reflects across x-axis.

Let's use that to build the general formula!

Step 0 Let θ be angle ℓ forms with horizontal (wherever you measure it), P a point on ℓ.



Step 1 Move P to O: $X - P$

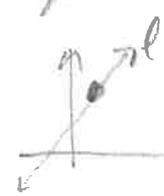


Step 2 Rotate by $-\theta$: $R_{-\theta}(X-P)$

Step 3 Reflect over x-axis: $\begin{bmatrix}1&0\\0&-1\end{bmatrix}R_{-\theta}(X-P)$

Step 4 Rotate back: $R_\theta\begin{bmatrix}1&0\\0&-1\end{bmatrix}R_{-\theta}(X-P)$

Step 5 Move O back to P:

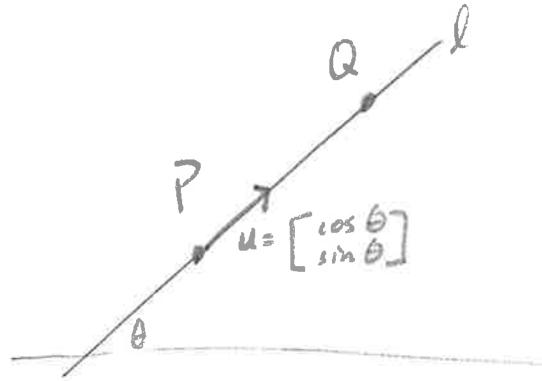


$$\underbrace{R_\theta\begin{bmatrix}1&0\\0&-1\end{bmatrix}R_{-\theta}}_{\text{by computation, this is } F_\theta}(X-P)+P$$

$$\Rightarrow \mathcal{M}_\ell(X) = F_\theta(X-P)+P$$

(Do an example w/ specific line)

⚠ We must verify we get same formula if we use a different $Q \in \ell$:



Let $u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, which is unit DI of ℓ

By warmup problem/computation, $\boxed{F_\theta u = u}$. Then:

$$F_\theta(X - Q) + Q = F_\theta(X \underbrace{-P+P}_{\text{clever addition of } 0!} - Q) + Q$$

$$= F_\theta(X - P + \underbrace{(P-Q)}_{= tu,\ \text{some } t}) + Q$$

$$= F_\theta(X-P) + F_\theta(tu) + Q$$

$$= F_\theta(X-P) + t\boxed{F_\theta u} + Q$$

$$= F_\theta(X-P) + tu + Q$$

$$= F_\theta(X-P) + (P-Q) + Q$$

$$= F_\theta(X-P) + P$$

Huzzah!

Now that we have reflections and rotations, it's time to develop some techniques/examples/properties.

Ex What is $\mathcal{R}_{\theta,c} \circ \mathcal{R}_{\varphi,c}$? (Successive rot'n by θ, φ centered at c):

$$R_\theta\left(\left[R_\varphi(x-c)+c\right]-c\right)+c = R_\theta R_\varphi(x-c)+R_\theta c - R_\theta c + c$$

$$= \underbrace{R_{\theta+\varphi}}_{\text{by previous computation/warmup.}}(x-c)+c$$

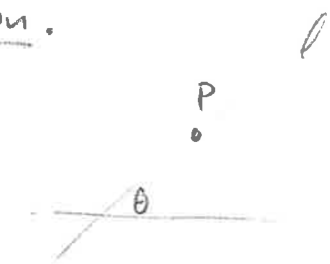
Ex What is $\mathcal{M}_k \circ \mathcal{M}_\ell$?

(do activity)

Answer: it depends on relationship b/w k, ℓ.

1. $k = \ell$
2. $k \parallel \ell$
3. $k \cap \ell = \{c\}$

① $M_\ell \circ M_\ell (X) = id(X)$, i.e. M_ℓ is an involution.



$$F_\theta\Big(\underbrace{[F_\theta(X-P)+P]}_{M_\ell(X)} - P\Big) + P$$

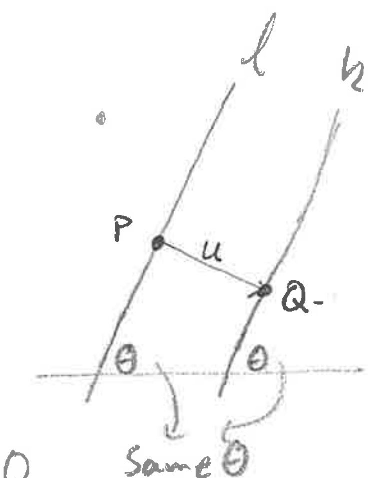
$$= \Big(\underbrace{F_\theta F_\theta}_{=I \text{ by computation/warmup}}(X-P) + \cancel{F_\theta P} - \cancel{F_\theta P}\Big) + P$$

$$= X - P + P$$

$$= X.$$

② $M_k \circ M_\ell (X) = T_{2u}(X)$

Choose $u = Q-P$, $Q \in k$, $P \in \ell$, $u \perp \ell$
(hence $u \perp k$ too)



$$M_k\big(F_\theta(X-P)+P\big) = F_\theta\Big(\big[F_\theta(X-P)+P\big] - Q\Big) + Q$$

$$= (X-P) + F_\theta P - F_\theta Q + Q$$

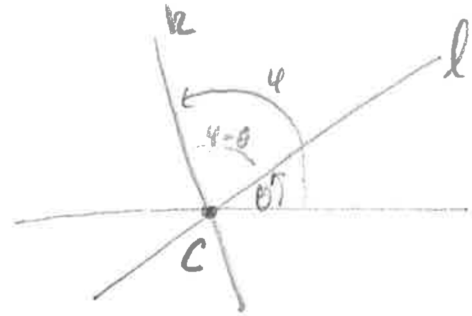
$$= X + (Q-P) - \underbrace{F_\theta(Q-P)}_{= -(Q-P) \text{ by computation/warmup.}}$$

$$= X + (Q-P) + (Q-P)$$

$$= X + 2(Q-P)$$

$$= X + 2u.$$

(3) $M_k \circ M_\ell(X) = R_{C,2(\varphi-\theta)}(X)$



$$M_k\left(F_\theta(X-C)+C\right) = F_\varphi\left[\left(F_\theta(X-C)+C\right)-C\right]+C$$

$$= F_\varphi F_\theta(X-C)+C$$

$$= R_{2(\varphi-\theta)}(X-C)+C.$$

Back to rotations: what about $R_{\varphi,D}\circ R_{\theta,C}$?

Case 1 $\theta=\varphi=\pi$. Looks like a translation!

$$R_{\pi,D}\left(\left[R_\pi(X-C)+C\right]\right) = \ell_D\left(\ell_C(X)\right)$$

$$= \ell_D(2C-X)$$

$$= 2D-2C+X$$

$$= X+2(D-C)$$

Case 2 general case (write out - it's hard!) cut the knot demo

Rather than test/try every possible combination of T, R, M, need a systematic approach. Key is...

Thm 6.16 Every isometry can be expressed (constructed) as the composition of ≤ 3 reflections.

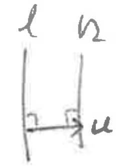
"Pf" Lab!

$\Rightarrow$ we just need to figure out all possibilities for $n=1, 2,$ or 3 reflections!

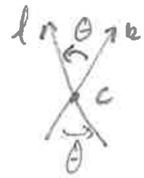
$n=1$ M_ℓ reflection

$n=2$ $M_\ell \circ M_k$ is either:

- $T_{2u}(X)$ if $\ell \parallel k$



- $R_{2\theta, C}$ if $\ell \cap k = \{C\}$



special cases: • if $\ell = k$, $M_\ell \circ M_k = id(X) = T_0(X) = R_{0,C}(X)$.

• if $\ell \perp$, $M_\ell \circ M_k = \ell_C(X)$.

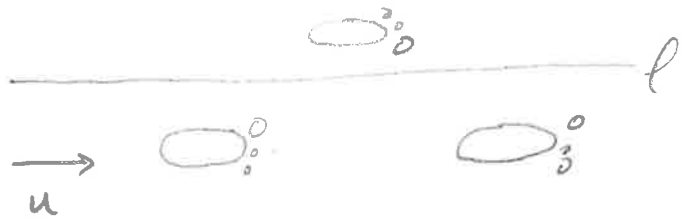
$n=3$ $M_k \circ M_\ell \circ M_m = ?$

$\exists$ new possibility... a glide reflection!

<u>Def</u> Given $u \parallel \ell$, $\mathcal{G}(x) = \mathcal{G}(x) = M_\ell \circ T_u(x)$

is a <u>glide reflection</u>

<u>Ex</u>



etc.

<u>Prop</u> Given $u \parallel \ell$, $M_\ell \circ T_u = T_u \circ M_\ell$

(so can do transl'n / refl'n in either order)

<u>Pf</u>: $M_\ell \circ T_u(x) = F_\theta\big([x+u]-P\big)+P$

$= \dots$

$= F_\theta(x-P) + P + F_\theta u$

$= \big(\ F_\theta(x-P)+P\ \big) + u$

$= T_u \circ M_\ell(x)$.

(Do Glide Refl'ns activity)

Escher Pictures

How do we know $\mathcal{G}(x)$ isn't just a reflection in hiding? Key turns out to be fixed points...

Fixed points: $f(x) = x$. $[\, \tau_c(X) = X$ on HW... $]$

Prop if $U \neq 0$, $\mathcal{T}_U(X)$ has no fixed points:

Pf if $X + U = X$, $U = 0$.

Prop only fixed point of $\mathcal{R}_{\theta,C}(X)$ is C (unless $\theta = 0$)

Pf Suppose $X = R_\theta(X - C) + C$

$$(X - C) = R_\theta(X - C).$$

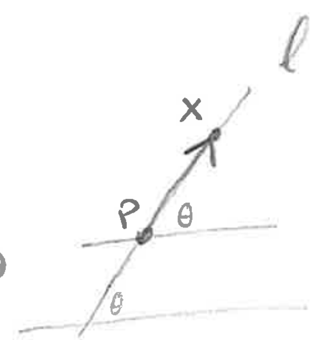
We know: R_θ changes every nonzero vector in $\mathbb{R}^2$ (unless $\theta = 0$) so $X - C = 0 \Rightarrow X = C$

Prop Fixed points of $\mathcal{M}_\ell(X)$ are ℓ

Pf Suppose $X = F_\theta(X - P) + P$

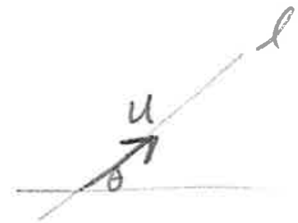
$$(X - P) = F_\theta(X - P)$$

We've seen: fixed points/vectors of F_θ are exactly those $\parallel \ell$. So $(X - P)$ DI of ℓ, and $P + (X - P) = X \in \ell$.



Prop $\mathcal{G}(X)$ is not a transl'n, rot'n, or a reflection; it's a "new" kind of isometry.

Pf $\mathcal{G}(X) = \mathcal{T}_u(\mathcal{M}_\ell(X)) = F_\theta(X-P)+P+u$



(note $F_\theta u = u$ by previous work...)

$\mathcal{G}(X) = F_\theta X + (\text{stuff})$ so its matrix is $F_\theta \Rightarrow$ it's not a rot'n or transl'n.

To show $\mathcal{G}(X)$ not a reflection, show it has no fixed points.

Suppose $X = \left(F_\theta(X-P)+P\right)+u$ ⊛

By ⊛, $\boxed{(X-P)-u = F_\theta(X-P)}$ (rearrange)

Also by ⊛, $F_\theta X = F_\theta F_\theta(X-P) + F_\theta P + F_\theta u$

$$F_\theta X = X - P + F_\theta P + u$$

$$\boxed{F_\theta(X-P) = (X-P)+u}$$

Comparing, we see this is only possible if $u=-u$, i.e. $u=0$, so $\mathcal{G}(X)$ would really be ... a reflection!

So as long as $u \neq 0$, $\mathcal{G}(X)$ not a refl'n.

Ok, so $\mathcal{G}(X)$ is a new kind of isometry, made up of 3 reflections. Are there any others I can get from 3 reflections?

$$\underbrace{\mathcal{M}_k \circ \overbrace{\mathcal{M}_\ell}}_{} \circ \mathcal{M}_m$$

either of these can be rewritten as a transl'n or rot'n (possibly even $\mathcal{I}(X)$...)

So we have to consider 4 possibilities

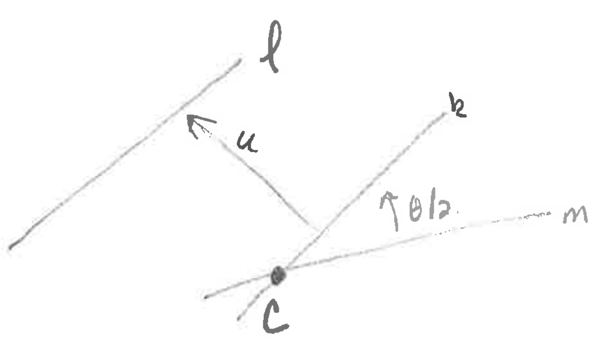
transl'n ∘ reflection
refl'n ∘ transl'n
} already know this is GR **if** transl'n $\|$ mirror.

rot'n ∘ refl'n

refl'n ∘ rot'n

Consider $\mathcal{M}_\ell \circ \mathcal{R}_{\theta,C}$. $\mathcal{R}_{\theta,C}$ constructed by reflecting across two lines ∩'ing at C, forming angle of $\underline{\theta/2}$.

I can choose lines so that the second is $\| \ell$:

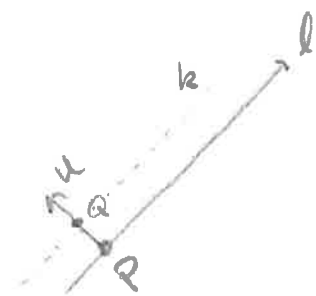


$$\mathcal{M}_\ell \circ \mathcal{R}_{\theta,C} = \mathcal{M}_\ell \circ (\mathcal{M}_k \circ \mathcal{M}_m)$$
$$= (\mathcal{M}_\ell \circ \mathcal{M}_k) \circ \mathcal{M}_m$$
$$= \mathcal{T}_{2u} \circ \mathcal{M}_m$$

(Similar argument for $R_{\theta,c} \circ M_\ell$). Thus we only need to consider composition of reflection and transl'n, in either order, $T_u \circ M_\ell$ or $M_\ell \circ T_u$. If $u \parallel \ell$ this is glide refl'n. What if $u \nparallel \ell$?

1st Case: $u \perp \ell$. (Explore with GeoGebra)

Prop: Suppose $u \perp \ell$. Then $T_u \circ M_\ell = M_k$, where $k = T_{u/2}(\ell)$.



$$
\begin{aligned}
T_u \circ M_\ell(X) &= T_u\left(F_\theta(X-P)+P\right)\\
&= F_\theta(X-P)+P+u\\
&= F_\theta(X-P)+P+\tfrac{1}{2}u+\underbrace{\tfrac{1}{2}u}_{=-\frac{1}{2}F_\theta,}\\
&= F_\theta\left(X-P-\tfrac{1}{2}u\right)+P+\tfrac{1}{2}u \qquad \text{since } F_\theta u = -F_\theta u \text{ here}\\
&= F_\theta(X-Q)+Q,
\end{aligned}
$$

for $Q = P + \frac{1}{2}u$

That's $M_k(X)$ for line k containing Q, same slope as ℓ.