Math 5335
Fall 2017
Exam 1
10/25/17
Time Limit: 75 Minutes

Name (Print): Solutions

This exam contains 7 pages (including this cover page) and 9 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and **put your initials** on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem**. If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. **Clearly indicate when you have done this.**

Do not write in the table to the right.

Page	Points	Score
2	14	
3	18	
4	13	
5	22	
6	21	
7	12	
Total:	100	

You may use the following matrices and computations on the exam without defining or proving them.

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad F_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}, \qquad R_\varphi R_\theta = R_{\varphi+\theta}, \qquad F_\varphi F_\theta = R_{2(\varphi-\theta)}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta), \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

1. Let ℓ be the line $(5,12)\cdot X = 17$.

   (a) (4 points) Find a parametric representation of ℓ.

$(5,12)\cdot(1,1) = 17$, so $(1,1)$ is on line.

$(5,12) \perp \ell$, so $(-12,5) \parallel \ell$

Other pts / slopes:

$U = \pm(-12/5, 1)$

$P = (17/5, 0), (0, 17/12)$

$\ell = \{(1,1) + s(-12,5) : s \in \mathbb{R}\}$

(other answers possible)

   (b) (4 points) Find a special normal form of ℓ.

Need a unit length normal vector: $\dfrac{(5,12)}{\sqrt{25+144}} = \dfrac{(5,12)}{\sqrt{169}} = \left(\dfrac{5}{13}, \dfrac{12}{13}\right)$

So take eqn for ℓ, divide both sides by 13:

$\frac{1}{13}(5,12)\cdot X = 17$

$\left(\frac{5}{13}, \frac{12}{13}\right)\cdot X = \frac{17}{13}$

Other answers possible, e.g.

$\left(\frac{5}{13}, \frac{12}{13}\right)\cdot\left(X - (1,1)\right) = 0.$

2. (6 points) Prove that $\overline{PQ} = \{aP + bQ \mid a+b=1, a,b \geq 0\}$.

We know $\overline{PQ} = \{P + s(Q-P) \mid 0 \leq s \leq 1\}$

$= \{(1-s)P + sQ \mid 0 \leq s \leq 1\}$

$= \{aP + bQ \mid a = 1-b,\ 0 \leq b \leq 1\}$ $b=s,\ a=1-s$

$= \{aP + bQ \mid a+b=1,\ a,b \geq 0\}$ if $a=1-b$ and $0 \leq b \leq 1$, $a \in [0,1]$ too.

3. Recall that $\arccos z = \int_z^1 \frac{1}{\sqrt{1-t^2}} dt$ and we define $\pi = \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} dt$.

(a) (6 points) Let rays p and q emanate from a common vertex, with direction indicators $(1,3)$ and $(2,-1)$. Find $|\angle(p,q)|$. An answer with an integral is fine.

$$U \cdot V = \frac{(1,3)}{\sqrt{10}} \cdot \frac{(2,-1)}{\sqrt{5}} = \frac{-1}{\sqrt{50}} = \frac{-1}{2\sqrt{5}}$$

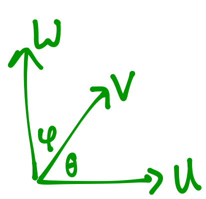
$$|\angle(p,q)| = \int_{-1/2\sqrt{5}}^{1} \frac{1}{\sqrt{1-t^2}} dt$$

(b) (6 points) Use calculus to prove that $\arccos 0 = \pi/2$.

$$\arccos 0 = \int_0^1 \underbrace{\frac{1}{\sqrt{1-t^2}}}_{\text{even fn}} dt = \frac{1}{2}\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} dt = \frac{\pi}{2}$$

(c) (6 points) Prove $\arccos \frac{1}{\sqrt{2}} = \pi/4$ using the methods of this course.

Let $U=(1,0)$, $V=(1/\sqrt{2}, 1/\sqrt{2})$, $W=(0,1)$ be unit DI's for angles with measure θ, φ as shown. Then $\theta = \varphi$, since



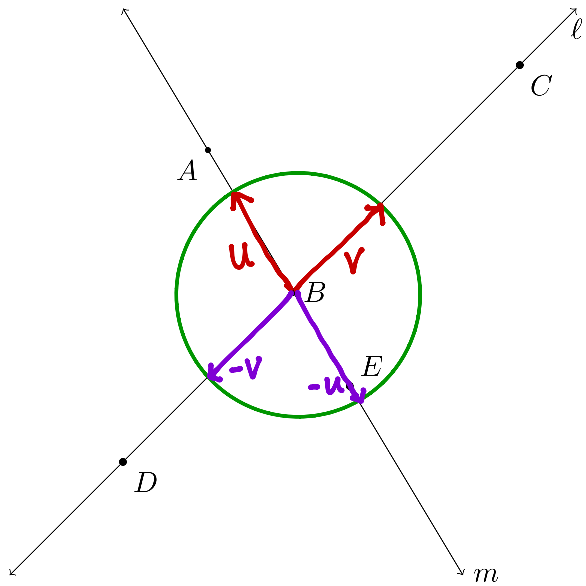
$$\left.\begin{aligned} \theta &= \arccos(U\cdot V) = \arccos(1/\sqrt{2}) \\ \varphi &= \arccos(V\cdot W) = \arccos(1/\sqrt{2}) \end{aligned}\right\} \text{same}$$

We also see $\theta + \varphi = \arccos(U \cdot W) = \arccos(0) = \pi/2$

Thus $\theta + \varphi = 2\theta = \pi/2$

$\Rightarrow \theta = \arccos(1/\sqrt{2}) = \pi/4.$

4. (8 points) Prove $|\angle ABC| = |\angle DBE|$ using the definitions and methods of this course.



Let $U = \frac{A-B}{||A-B||}$, $V = \frac{C-B}{||C-B||}$. Then U, V are unit DI's for $|\angle ABC|$, and $-U, -V$ are unit DI's for $|\angle DBE|$. Thus

$$|\angle ABC| = \arccos(U \cdot V)$$

$$|\angle DBE| = \arccos((-U)\cdot(-V))$$

$$= \arccos(U \cdot V)$$

$$= |\angle ABC|,$$

as desired.

5. (5 points) Given two vectors U and V, prove $||U + V||^2 = ||U||^2 + ||V||^2 + 2\,U \cdot V$ using methods from class.

$$||U+V||^2 = (U+V)\cdot(U+V)$$

$$= U\cdot U + U\cdot V + V\cdot U + V\cdot V$$

$$= ||U||^2 + ||V||^2 + 2U\cdot V$$

6. (a) (4 points) Complete the definition: $\mathcal{U}(X)$ is an isometry of $\mathbb{R}^2$ if...

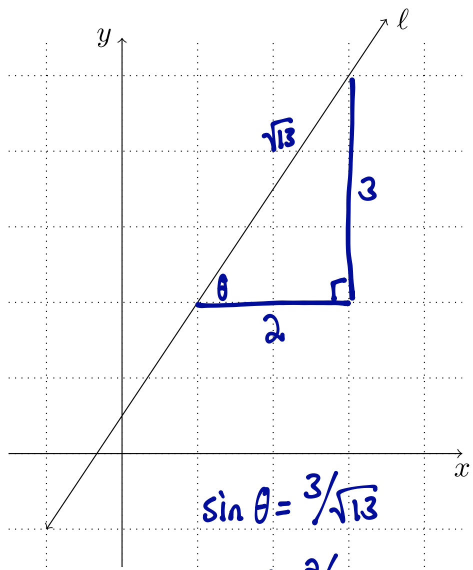
$$\|\mathcal{U}(P)-\mathcal{U}(Q)\| = \|P-Q\| \quad \forall\ P, Q \in \mathbb{R}^2$$

(b) (6 points) Let $\mathcal{U}$ and $\mathcal{V}$ be isometries of $\mathbb{R}^2$. Prove that the composition $\mathcal{U} \circ \mathcal{V}$ is an isometry.

$$\|\mathcal{U}(\mathcal{V}(P)) - \mathcal{U}(\mathcal{V}(Q))\| = \|\mathcal{V}(P)-\mathcal{V}(Q)\| \quad \text{b/c } \mathcal{U} \text{ is an isometry}$$

$$= \|P-Q\| \quad \text{b/c } \mathcal{V} \text{ is an isometry}$$

(c) (12 points) Let $\ell = \{(1,2) + t(2,3)\}$, where $t \in \mathbb{R}$. Find the matrix formula $\mathcal{M}_\ell(X)$ for the reflection across the line ℓ. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.



$\sin\theta = 3/\sqrt{13}$

$\cos\theta = 2/\sqrt{13}$

$$\mathcal{M}_\ell(X) = F_\theta(X-P)+P$$

$$= \begin{bmatrix} -5/13 & 12/13 \\ 12/13 & 5/13 \end{bmatrix}\left(X - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\cos 2\theta = \frac{4}{13} - \frac{9}{13} = -\frac{5}{13}$$

$$\sin 2\theta = 2 \cdot \frac{3 \cdot 2}{13} = \frac{12}{13}$$

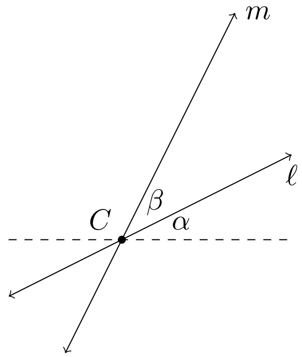
7. (10 points) Find the matrix formula $\mathcal{R}(X)$ for the rotation by $2\pi/3 = 120°$ centered at the point $(-3, 4)$. Your answer should include exact values in the matrix, not trig functions, but you do not need to multiply out your entire formula.

$$\mathcal{R}(X) = R_\theta(X - C) + C = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}\left(X - \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right) + \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$R_{2\pi/3} = \begin{bmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

8. Suppose lines ℓ and m intersect at C, with angles α and β as shown below.

(a) (4 points) Write down matrix formulas for $\mathcal{M}_\ell(X)$ and $\mathcal{M}_m(X)$, the reflections across these lines. Make sure to define and label (on the diagram) any new letters or notation you introduce, such as θ.



$$\mathcal{M}_\ell(X) = F_\alpha(X - C) + C$$

$$\mathcal{M}_m(X) = F_{(\alpha+\beta)}(X - C) + C$$

(b) (7 points) Now consider the composition $\mathcal{M}_m \circ \mathcal{M}_\ell(X) = \mathcal{M}_m(\mathcal{M}_\ell(X))$. Use the methods of this course (and your answers to the previous part) to identify the resulting isometry. If it is a translation, what is the translation vector? If it is a rotation, by how much, and centered at which point? If it is a reflection, what is the mirror? Citing the answer from memory will result in a few points; for full credit you must justify your answer.

$$\mathcal{M}_m \circ \mathcal{M}_\ell(X) = \mathcal{M}_m\left(F_\alpha(X - C) + C\right)$$

$$= F_{(\alpha+\beta)}\left(\left[F_\alpha(X - C) + \underbrace{C\right] - C}_{\text{cancel}}\right) + C$$

$$= R_{\underbrace{2(\alpha+\beta-\alpha)}_{\text{by given info on cover page}}}(X - C) + C$$

$$= R_{2\beta}(X - C) + C$$

i.e rotation by 2β, centered at C.

9. (12 points) Indicate whether each statement is **True** or **False** by circling the appropriate answer. Justify your answer with definitions, theorems and methods from this course. (If false, **be specific** with your explanation; e.g. tell me what part of a definition is not satisfied, or give an example to show the statement is false, etc.)

(a) For any A, B and C, the barycentric coordinates of the origin $(0,0)$ are always $(0,0,0)^{\triangle ABC}$.

True **False**

$(0,0,0)^{\triangle}$ not valid BC's, because $0+0+0 \neq 1$.

(b) If $\mathcal{U}(X)$ is an isometry, then $\mathcal{U}(aP+bQ) = a\mathcal{U}(P) + b\mathcal{U}(Q)$ for all $a, b \in \mathbb{R}$ and $P, Q \in \mathbb{R}^2$.

True **False**

Only valid in general if $a+b=1$.

Counter-example: $P=(0,0)$, $Q=(0,1)$, $\mathcal{U}(X) = X + (1,0)$, $a=b=1$

$\mathcal{U}(aP+bQ) = \mathcal{U}((0,1)) = (1,1)$

$1 \cdot \mathcal{U}(0,0) + 1 \cdot \mathcal{U}(0,1) = (1,0) + (1,1) = (2,1)$

(c) In a glide reflection, the "glide" is always perpendicular to the mirror line.

True **False**

The glide is $\parallel$ mirror, not $\perp$.