Math 5335 **Name (Print):** Solutions
Fall 2017
Exam 2
12/6/17
Time Limit: 75 Minutes

This exam contains 7 pages (including this cover page) and 11 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and **put your initials** on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you are applying a theorem, you should indicate this fact**, and explain why the theorem may be applied.
- **Do not trivialize a problem**. If you are asked to prove a theorem, you cannot just cite that theorem.
- **Organize your work** in a reasonable, tidy, and coherent way. Work that is disorganized and jumbled that lacks clear reasoning will receive little or no credit.
- **Unsupported answers will not receive full credit.** An answer must be supported by calculations, explanation, and/or algebraic work to receive full credit. Partial credit may be given to well-argued incorrect answers as well.
- If you need more space, use the back of the pages. **Clearly indicate when you have done this.**

Do not write in the table to the right.

Page	Points	Score
2	17	
3	15	
4	13	
5	20	
6	22	
7	13	
Total:	100	

You may use the following results on the exam without defining or proving them.

The distance between points (a, b) and (a, d) in the Poincaré Half Plane is $|\ln(d/b)|$.

The distance between points P_1 and P_2 on the line $(x-\omega)^2 + y^2 = \rho^2$, with angles $t_1 = |\angle(\omega+\rho, 0)(\omega, 0)P_1|$ and $t_2 = |\angle(\omega+\rho, 0)(\omega, 0)P_2|$ is

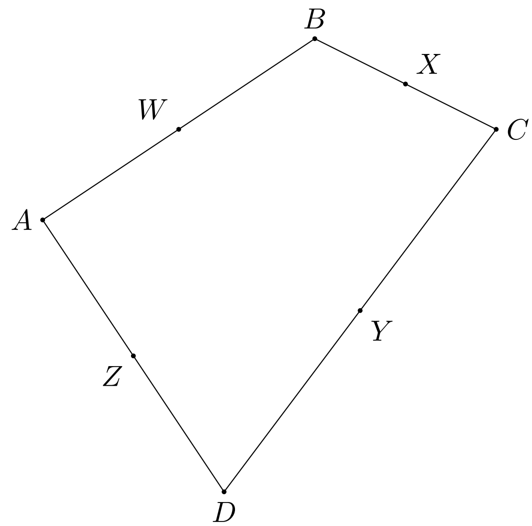
$$\ln\left[(\csc t_2 - \cot t_2) \,/\, (\csc t_1 - \cot t_1)\right]$$

1. Let $ABCD$ be quadrilateral which is simple, but not necessarily a parallelogram, trapezoid, or other familiar shape. Let W, X, Y and Z be the midpoints of the sides, as shown in the generic diagram below.

   (a) (4 points) What is W in terms of A and B? Write similar expressions for X, Y, and Z.

   $W = \frac{A+B}{2}$ $\quad X = \frac{B+C}{2}$ $\quad Y = \frac{C+D}{2}$ $\quad Z = \frac{A+D}{2}$

   (b) (8 points) Prove that $WXYZ$ is a parallelogram.



Many methods possible, e.g.:

$$W+Y = \frac{A+B+C+D}{2} = \frac{B+C+A+D}{2} = X+Z.$$

(Could show any of the other conditions in Thm 8.4)

2. (5 points) Find the area of quadrilateral $ABCD$ if $A = (-1, 0)$, $B = (5, 3)$, $C = (6, 7)$ and $D = (0, 4)$. If you use a formula, explain why it applies. (You can use the grid if it's helpful, or ignore it.)

$B-A = (5,3)-(-1,0) = (6,3) = C-D$

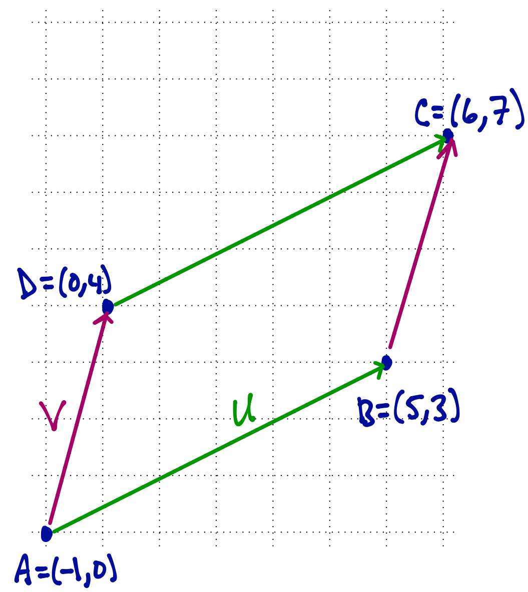
$D-A = (1,4) = C-B$

Thus $ABCD$ is a ||'gram and its area is

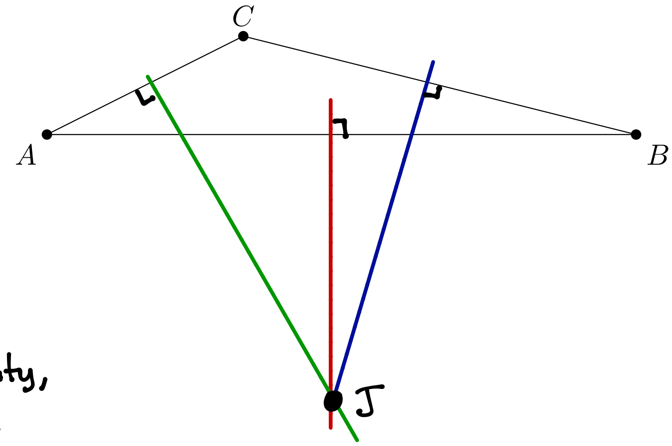
$$\left|\det\begin{bmatrix} u & v \end{bmatrix}\right| = \left|\det\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}\right|$$

$$= |24-3|$$

$$= 21$$



3. (5 points) Prove that the perpendicular bisectors of (the sides of) $\triangle ABC$ all intersect in a point J which is equidistant to all three vertices. (You can use the diagram below for convenience but your argument must apply to all triangles.)

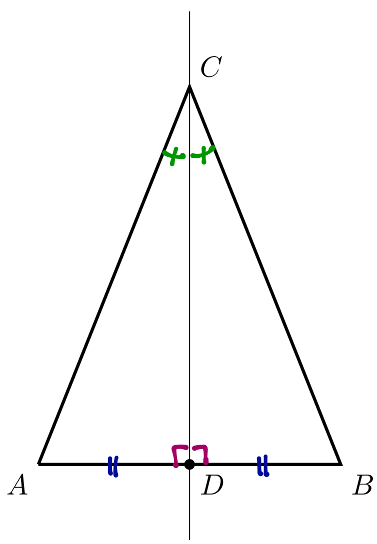


Let J be the intersection of $\perp$ bisectors of $\overline{AB}$ and $\overline{BC}$. (Because $\overline{AB}, \overline{BC}$ not $\parallel$, the $\perp$ bisectors aren't $\parallel$ either, hence intersect.)

Thus $|\overline{JA}| = |\overline{JB}|$ and $|\overline{JB}| = |\overline{JC}|$. By transitivity, $|\overline{JA}| = |\overline{JC}|$. Hence J is also on $\perp$ bisector of $\overline{AC}$.

4. A corollary to Theorem 7.16 in the book says that, unless $\triangle ABC$ is equilateral, there is a unique line through its centroid, orthocenter and circumcenter. This line is known as its *Euler Line.*

(a) (7 points) Let $\triangle ABC$ be isosceles, with $\overline{AC} \cong \overline{BC}$, as shown below. Prove the Euler Line of $\triangle ABC$ is the median from C, which intersects $\overline{AB}$ at D as shown.



D midpt of $\overline{AB} \Rightarrow \overline{AD} \cong \overline{BD}$. By SSS congruence, $\triangle ADC \cong \triangle BDC$. Thus

(1) $\angle ADC$ and $\angle BDC$ are congruent and combine to form a right angle, so each measures $\pi/2 = 90^\circ$. Thus $\overleftrightarrow{CD}$ is an altitude...

(2)... and, because $\overline{AD} \cong \overline{BD}$, $\overleftrightarrow{CD}$ is $\perp$ bisector, too.

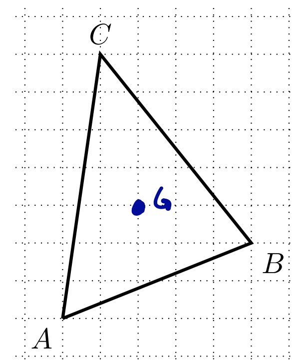
Thus $\overleftrightarrow{CD}$ is a median, $\perp$ bisector and altitude, so contains the centroid, circumcenter and orthocenter; thus it's the Euler line.

(b) (3 points) Why does an equilateral $\triangle ABC$ not have an Euler Line?

In an equilateral $\triangle ABC$, $G = H = J$, so those "three" points don't determine a line.

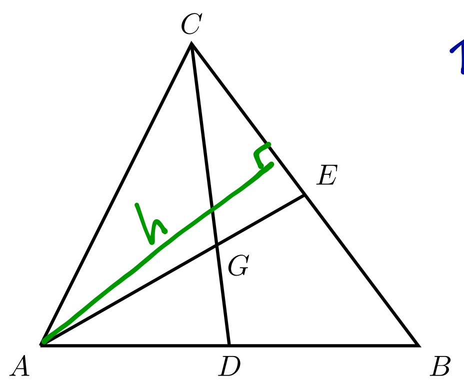
5. (5 points) Let $A = (0,0)$, $B = (5,2)$ and $C = (1,7)$. Find the centroid of $\triangle ABC$, in both barycentric and rectangular coordinates.

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^{\triangle ABC} = (2,3)$$



6. In $\triangle ABC$ below, $\overline{AE}$ and $\overline{CD}$ are medians, intersecting at the centroid G. Let $x = ||\triangle ABC||$, the area of the large triangle.

(a) (5 points) Explain why $||\triangle ACE|| = \frac{1}{2}x$.



Treating $\overleftrightarrow{BC}$ as the "base" line, $\triangle ABC$ and $\triangle AEC$ share an altitude from A to $\overleftrightarrow{BC}$, hence the same height. Since E is the midpoint of $\overline{BC}$,

$$||\triangle AEC|| = \frac{1}{2}|\overline{EC}|\cdot h = \frac{1}{2}\left(\frac{1}{2}|\overline{BC}|\right)h$$

$$= \frac{1}{2}\left(\frac{1}{2}|\overline{BC}|\cdot h\right)$$

$$= \frac{1}{2}x.$$

(b) (3 points) Find $||\triangle ACG||$ in terms of x. Justify your answer.

Because G is the centroid, it's 1/3 of way from E to A, so $|\overline{GE}| = 2|\overline{GA}|$.

Thus (treating $\overline{AE}$ and $\overline{GE}$ as bases, and an altitude from C),

$$||\triangle ACG|| = \frac{2}{3}||\triangle ACE|| = \frac{2}{3}\cdot\frac{1}{2}x = \frac{1}{3}x$$

(or can use all 3 medians to split $\triangle$ into 6 $\triangle$'s of equal area $\left(=\frac{x}{6}\right)$, as in HW, and combine two of them to form $\triangle ACG$ with area $\frac{2x}{6} = \frac{1}{3}x$.)

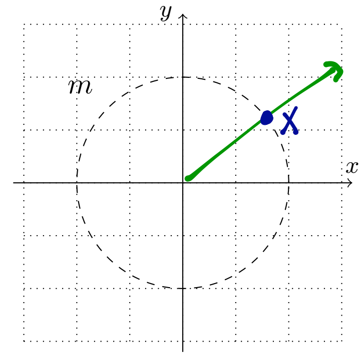
7. (a) (5 points) Let m be a circle centered at C with radius ρ. Prove: if $X \in m$, then $X' = X$, where X' is the reflection (or inversion) of X across m.

$X' \in \overrightarrow{OX}$ chosen such that (*)

$|\overline{OX}| \cdot |\overline{OX'}| = \rho^2$

$\rho\, |\overline{OX'}| = \rho^2$

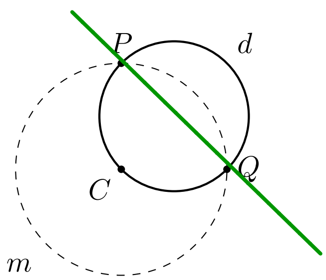
$|\overline{OX'}| = \rho$



The only possibility is $X' = X$.

(without (*), we only know $X' \in m$, not necessarily $X' = X$.)

(b) (5 points) Let m be a circle centered at C, and suppose d is a circle containing C which intersects m at points P and Q as shown. Sketch d', the reflection of d across m. Briefly justify your answer.

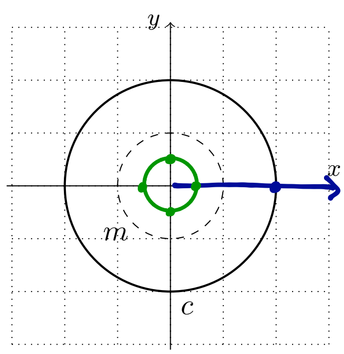


$C \in d \Rightarrow \infty \in d' \Rightarrow d'$ is a line.

$P, Q \in m$ are fixed, so $P, Q \in d'$

Thus $d' = \overleftrightarrow{PQ}$

(c) (5 points) Let m be the circle $x^2 + y^2 = 1$ and c the circle $x^2 + y^2 = 4$. Sketch and find an equation (with brief justification) for c', the reflection of c across m.



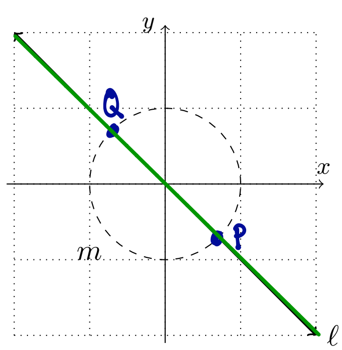
$X = (2,0) \in c$ sent to $X' \in \overrightarrow{OX}$ (= pos. x-axis) s.t.

$\|X\| \cdot \|X'\| = 1 \Rightarrow 2\|X'\| = 1 \Rightarrow \|X'\| = \frac{1}{2} \Rightarrow X' = (\frac{1}{2}, 0)$

Similarly, $(0, \pm 2) \mapsto (0, \pm \frac{1}{2})$, $(-2, 0) \mapsto (-\frac{1}{2}, 0)$

Thus c': $x^2 + y^2 = (\frac{1}{2})^2$

(d) (5 points) Let m be the circle $x^2 + y^2 = 1$ and ℓ the line $y = -x$. Sketch and find an equation (with brief justification) for ℓ', the reflection of ℓ across m.



$0 \in \ell \Rightarrow \infty \in \ell'$ (so ℓ' is a line)

$\infty \in \ell \Rightarrow 0 \in \ell'$

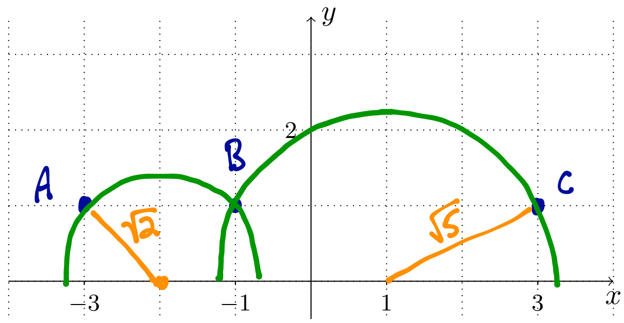
$P, Q \in \ell$ are on m, so $P, Q \in \ell'$

Thus ℓ' is also the line $y = x$

On this page, all points, lines, segments and distances are in the **Poincaré Half Plane**.

8. Let $A = (-3, 1)$, $B = (-1, 1)$ and $C = (3, 1)$.

   (a) (9 points) Sketch the (Poincaré) lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$. Find the equation for each line.



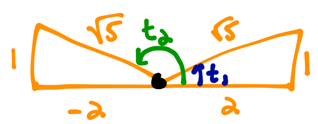
$\overleftrightarrow{AB}$ centered at $(-2,0)$, radius $\sqrt{2}$

$(x+2)^2+y^2=2$

$\overleftrightarrow{BC}$ centered at $(1,0)$, radius $\sqrt{5}$

$(x-1)^2+y^2=5$

   (b) (5 points) Find the length of segment $\overline{BC}$. (For full credit, your final answer shouldn't contain any trig functions, but you don't have to evaluate any logarithms.)



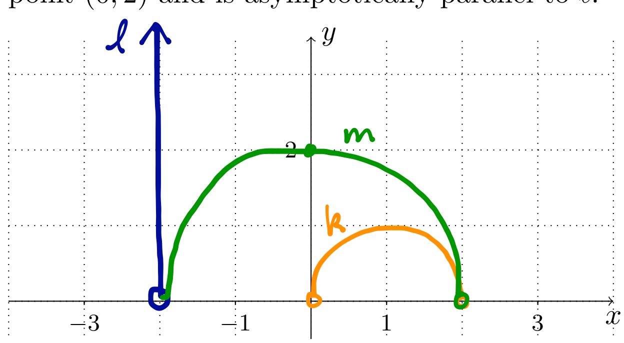
Using formula on cover page,

$$|\overline{BC}| = \left| \ln \frac{\csc t_2 - \cot t_2}{\csc t_1 - \cot t_1} \right| = \left| \ln \frac{\sqrt{5}+2}{\sqrt{5}-2} \right|$$

(Can use Thm 11.1 if you memorized it.)

9. Let ℓ be the line directed by $\mathfrak{A} = (-2, 0)$ and $\mathfrak{B} = (\infty, 0)$.

   (a) (5 points) Sketch ℓ on the grid below. Then sketch and give an equation for a line m which contains the point $(0, 2)$ and is asymptotically parallel to ℓ.

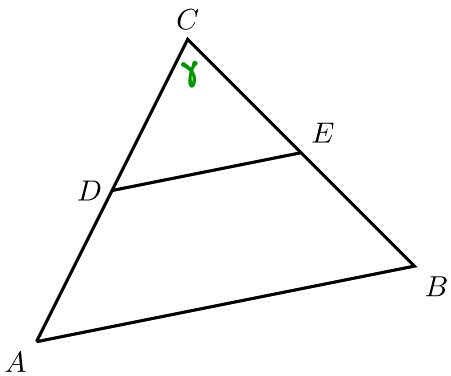


$m: x^2+y^2=4$ (or $x=0$ works, too!)

   (b) (3 points) Give an equation for a line k containing $(1, 1)$ which is ultra parallel to ℓ, or explain why none exists.

$(x-1)^2+y^2=1$

10. (5 points) Given $\triangle ABC$, let D and E be the midpoints of $\overline{AC}$ and $\overline{BC}$ as shown. Using any appropriate methods from the course, prove $\overline{DE} \parallel \overline{AB}$ and $|\overline{DE}| = \frac{1}{2}|\overline{AB}|$



Method 2

Because D, E are midpoints of $\overline{AC}, \overline{BC}$, we have $\frac{|\overline{AC}|}{|\overline{DC}|} = \frac{|\overline{EC}|}{|\overline{BC}|} = 2$.

Note also that $\triangle ACB$ and $\triangle BCE$ share the angle γ. Thus $\triangle ACB \sim \triangle DCE$ by SAS Similarity (#2 on previous page). That means

(a) $\frac{|\overline{AB}|}{|\overline{DE}|}$ is also 2, as needed, and

(b) $\angle CDE \cong \angle CAB \Rightarrow \overline{DE} \parallel \overline{AB}$ (corresponding angles)

Method 1

$$E - D = \frac{B+C}{2} - \frac{A+C}{2} = \frac{1}{2}(B-A)$$

Thus $E-D \parallel B-A$ and $E-D$ is half as long as $B-A$.

11. For each of the following statements, indicate whether it is **True** or **False** by circling the corresponding answer. Briefly justify your answer.

(a) (4 points) Let $ABCD$ be a simple quadrilateral. If $A + B = C + D$, then $ABCD$ is a parallelogram.

True **False**

(The condition is $B + D = A + C$.)

(b) (4 points) If $\triangle DEF$ is the image of $\triangle ABC$ under a conformal affinity, then their areas must be equal.

True **False**

Conformal affinities can scale $\mathbb{R}^2$, changing the area

(Think: similarity)