Warmup Problems:

For $R_\theta = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$, $F_\theta = \begin{bmatrix} c2\theta & s2\theta \\ s2\theta & -c2\theta \end{bmatrix}$, prove:

(a) $F_\theta \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

(Hence F_θ fixes any multiple of $\begin{bmatrix} c\theta \\ s\theta \end{bmatrix}$, too)

(b) $F_\theta F_\theta = I$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $F_\theta F_\varphi = \underline{R}_{2(\theta-\varphi)}$

Proof of (a)

- $$F_\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} c2\theta\, c\theta + s\theta\, \overbrace{s2\theta}^{\sin 2\theta = 2s\theta c\theta} \\ c\theta\, s2\theta - s\theta\, c2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\left(\overbrace{\cos 2\theta}^{c^2\theta - s^2\theta} + 2\sin^2\theta\right) \\ \sin\theta\left(2\cos^2\theta - \cos 2\theta\right) \end{bmatrix}$$

$$= \begin{bmatrix} c\theta\left(c^2\theta + s^2\theta\right) \\ s\theta\left(c^2\theta + s^2\theta\right) \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Isometry Day!

Goal: What isometries are possible? Have form

$$\mathcal{U}(X) = MX + P$$

where $M \in \{R_\theta, F_\theta\}$.

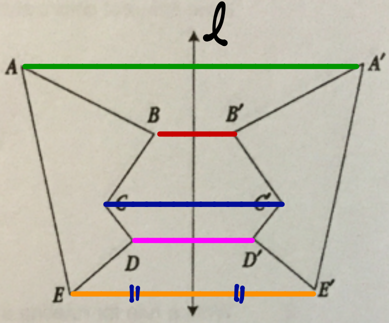
(I) Identity $\mathcal{U}(X) = X$ $\left(= IX + 0 = R_0 X + 0 \right)$

(II) $\mathcal{T}_V(X) = X + V$ $\left(= IX + V = R_0 X + V \right)$

Reflections

1. Draw the line segment connecting each point to its image.

ℓ is perp. bisector of $\overline{AA'}, \ldots, \overline{EE'}$.



2. Write your own rule for reflecting a point or object over a line.

For pt P: Find $k \perp \ell$, $P \in k$. P' is point on k on other side of ℓ, but same dist. from ℓ.

For object: reflect every pt in object.

3. Use your rule to reflect pentagon JKLMN over the line.

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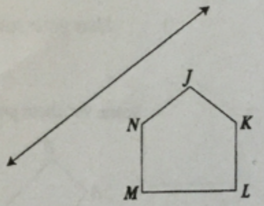
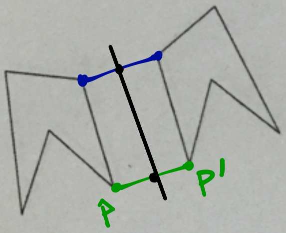


4. Find the line of reflection for the picture at right.

Draw $\perp$ bisector of $\overline{PP'}$



Rotations

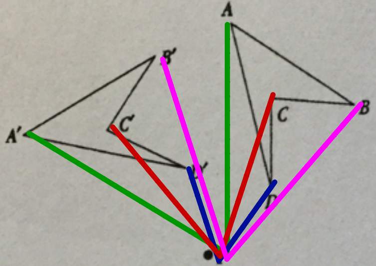
1. Pick a color and draw a line segment connecting A to P and A' to P. Pick three different colors and repeat with B, C, and D. What do you notice?

$\angle APA' \cong \angle BPB' \cong \angle CPC' \cong \angle DPD'$

$AP \cong A'P$

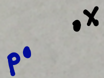
$\vdots$

$PD \cong PD'$

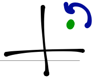


2. Write a rule for rotating a point or object about another point. P by angle θ.

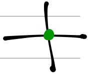
Construct X' to form $\angle XPX'$ with measure θ (+/- matters), $PX \cong PX'$.



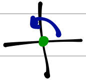
III) Rotation by θ about C: $R_{\theta,C}(X) = R_\theta(X-C)+C$



Step 1 Move C to O: $X - C$



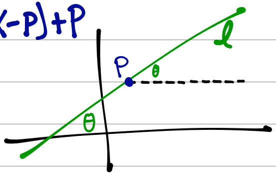
Step 2 Rotate by θ about O: $R_\theta(X-C)$



Step 3 Move back: $R_\theta(X-C)+C$

$\big(= MX + P$ for $M = R_\theta$,
$P = -R_\theta C + C\big)$

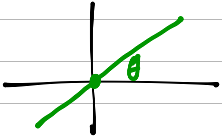
IV Reflection across ℓ: $M_\ell(X) = F_\theta(X-P)+P$



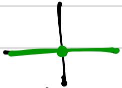
ℓ forms angle of θ w/ horizontal

$P \in \ell$

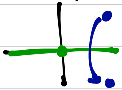
Step 1 Move P to 0: $(X-P)$



Step 2 Rotate new line to x-axis: $R_{-\theta}(X-P)$



Step 3 Reflect across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta}(X-P)$



Step 4 Rotate by $+\theta$: $R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta}(X-P)$

Step 5 Move line back: $\underbrace{R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta}}_{=F_\theta \text{ (next page)}}(X-P) + P$

$$R_\theta \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} R_{-\theta} = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}}$$

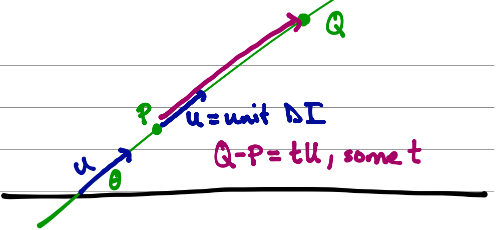
$$= \dots.$$

$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$= F_\theta$$

⚠ Our formula

$$M_\ell(X) = F_\theta(X-P) + P$$



depends on choice of P

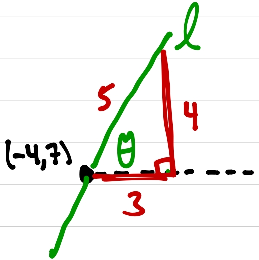
We need to verify $F_\theta(X-Q)+Q$ gives same formula $\forall\, Q \in \ell$.

$$\begin{aligned}
F_\theta(X-Q)+Q &= F_\theta(X - P + P - Q) + Q \\
&= F_\theta(X-P-(Q-P)) + Q \\
&= F_\theta(X-P-tU) + Q \\
&= F_\theta(X-P) - \underbrace{F_\theta(tU)}_{\substack{= tF_\theta U = tU \text{ by Prop} \\ = Q-P}} + Q \\
&= F_\theta(X-P) - Q + P + Q \\
&= F_\theta(X-P) + P.
\end{aligned}$$

Ex Find formula for refl'n across ℓ: $(-4,7)+t(3,4)$

$$M_\ell(x) = F_\theta\left(x - \begin{bmatrix}-4\\7\end{bmatrix}\right) + \begin{bmatrix}-4\\7\end{bmatrix}$$

$$F_\theta = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$$



$\cos\theta = 3/5$

$\sin\theta = 4/5$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9-16}{25} = -7/25$$

$$\sin 2\theta = 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}$$

$$M_\ell(x) = \begin{bmatrix}-7/25 & 24/25\\ 24/25 & 7/25\end{bmatrix}\left(x - \begin{bmatrix}-4\\7\end{bmatrix}\right) + \begin{bmatrix}-4\\7\end{bmatrix}$$

4 @ 5 pts

4 compl'n

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