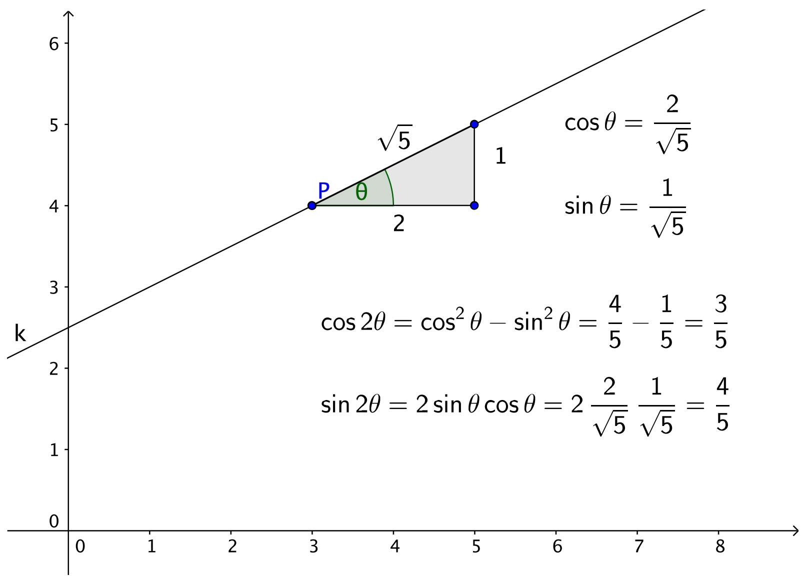
In class we constructed the matrix formula for a reflection across the line $\ell = \{(2,3) + s(1,\sqrt{3})\}$, where we could recognize that the line formed an angle of $\theta = \pi/3$ with the horizontal. On the homework you're asked to find a formula for reflecting across a line where that angle is not so evident. You can still find the matrix F_θ without resorting to a calculator, though, if you draw a nice picture and use trig identities. To show you how this works, here's how we can find the matrix formula for a reflection across the line $k = \{(3,4) + t(2,1)\}$ – i.e. the line through $P = (3,4)$ with direction indicator $U = (2,1)$. The picture below shows both the line and how to find $\cos\theta$, $\sin\theta$, $\cos 2\theta$ and $\sin 2\theta$, where θ is the angle formed by k and the horizontal.



Using the formula defined in class (together with the trig values found above),

$$\begin{aligned}\mathcal{M}_k(X) &= F_\theta(X - P) + P \\ &= \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\left(X - \begin{bmatrix}3\\4\end{bmatrix}\right) + \begin{bmatrix}3\\4\end{bmatrix} \\ &= \begin{bmatrix}3/5 & 4/5 \\ 4/5 & -3/5\end{bmatrix}\left(X - \begin{bmatrix}3\\4\end{bmatrix}\right) + \begin{bmatrix}3\\4\end{bmatrix}\end{aligned}$$

Normally in class I've suggested that the above forms are more useful in terms of recognizing what the isometry is, but sometimes you might want to multiply things out further:

$$\begin{aligned}\mathcal{M}_k(X) &= \begin{bmatrix}3/5 & 4/5 \\ 4/5 & -3/5\end{bmatrix} X - \begin{bmatrix}3/5 & 4/5 \\ 4/5 & -3/5\end{bmatrix}\begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} \\ &= \begin{bmatrix}3/5 & 4/5 \\ 4/5 & -3/5\end{bmatrix} X - \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}3\\4\end{bmatrix} \\ &= \begin{bmatrix}3/5 & 4/5 \\ 4/5 & -3/5\end{bmatrix} X + \begin{bmatrix}-2\\4\end{bmatrix}\end{aligned}$$

When is this form useful? Two situations come to mind. (1) If you want to actually use the formula to reflect various points, then this version is a little more compact. (2) If you want to compare two reflections (or isometries in general) to see if they're the same or different, then it's worth simplifying things down to the form $MX + P$.

(Remember that every isometry can be written in the form $\mathcal{U}(X) = MX + P$ for a *unique* choice of orthogonal matrix M and point P.) That's why I asked you to simplify the answers to homework problems 3.12 and 3.13 to this form.

If you want to check that this formula is correct, you could compute the reflection of various points across the line. For example:

$$\mathcal{M}_k\left(\begin{bmatrix}5\\0\end{bmatrix}\right) = \begin{bmatrix}3/5 & 4/5\\4/5 & -3/5\end{bmatrix}\begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}-2\\4\end{bmatrix} = \cdots = \begin{bmatrix}1\\8\end{bmatrix}$$
$$\mathcal{M}_k\left(\begin{bmatrix}0\\5\end{bmatrix}\right) = \begin{bmatrix}3/5 & 4/5\\4/5 & -3/5\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix} + \begin{bmatrix}-2\\4\end{bmatrix} = \cdots = \begin{bmatrix}2\\1\end{bmatrix}$$

You can check on the picture above that the reflection of $(5, 0)$ across k is in fact $(1, 8)$, and the reflection of $(0, 5)$ is $(2, 1)$. One further note: occasionally we might write the vector X in terms of its components x and y, in which case the formula takes the form:

$$\mathcal{M}_k\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}3/5 & 4/5\\4/5 & -3/5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}-2\\4\end{bmatrix} = \cdots = \begin{bmatrix}\frac{3}{5}x + \frac{4}{5}y - 2\\\frac{4}{5}x - \frac{3}{5}y + 4\end{bmatrix}$$

Or, writing our vectors out with parentheses instead:

$$\mathcal{M}_k(x, y) = \left(\frac{3}{5}x + \frac{4}{5}y - 2, \frac{4}{5}x - \frac{3}{5}y + 4\right)$$