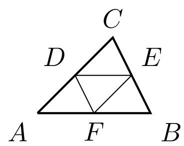
The goal of this handout is to prove *Varignon's Theorem*, which says the area of a convex quadrangle $ABCD$ is twice the area of the parallelogram with vertices at the midpoints of the side $ABCD$. Our proof will depend on the following facts about a triangle with the midpoints connected.



Theorem. Let D and E be the midpoints of $\overline{AC}$ and $\overline{BC}$. Then $\overline{DE} \parallel \overline{AB}$ and $|\overline{DE}| = \frac{1}{2}|\overline{AB}|$.

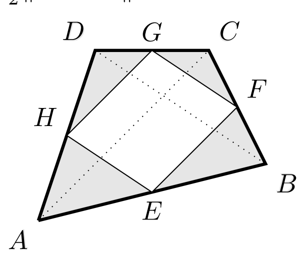
Proof: We proved this in class using SAS Similarity; solutions are posted on the course website in the folder with the various activities I've handed out during the semester. □

Corollary. The four triangles in the diagram above all have area $\frac{1}{4}\|\triangle ABC\|$.

Proof: By the theorem, the base $\overline{AB}$ of $\triangle ABC$ is twice as long as the base $\overline{DE}$ of $\triangle DEC$. By similarity, the height of $\triangle ABC$ is twice that of $\triangle DEC$ as well. Hence $\|\triangle ABC\| = 4\|\triangle DEC\|$, or $\|\triangle DEC\| = \frac{1}{4}\|\triangle ABC\|$. Now you can apply the theorem using the other sides of $\triangle ABC$ as the base, and show $\|\triangle EFB\| = \|\triangle FDA\| = \frac{1}{4}\|\triangle ABC\|$ as well. Overall, $\triangle ABC$ is divided into 4 triangles in our diagram. We've shown three of them have area $\frac{1}{4}\|\triangle ABC\|$. Hence the remaining triangle must have the same area. □

Prove. The area of a convex quadrangle $ABCD$ is twice the area of the parallelogram with vertices at the midpoints of the side $ABCD$.

Proof: In the diagram below, E, F, G and H are the midpoints of the sides of $ABCD$. The dotted lines are the diagonals of $ABCD$. We'll show $\|EFGH\| = \frac{1}{2}\|ABCD\|$.



The area of the parallelogram $EFGH$ is the area of $ABCD$ minus the area of the shaded triangles.

$$\|EFGH\| = \|ABCD\| - \|\triangle AEH\| - \|\triangle BFE\| - \|\triangle CGH\| - \|\triangle DHG\|$$

By the Corollary above, each shaded triangle is 1/4 the area of a larger triangle formed with one of the diagonals. For example, $\|\triangle AEH\| = \frac{1}{4}\|\triangle ABD$. Hence:

$$\begin{aligned}\|EFGH\| &= \|ABCD\| - \frac{1}{4}\|\triangle ABD\| - \frac{1}{4}\|\triangle BCA\| - \frac{1}{4}\|\triangle CDB\| - \frac{1}{4}\|\triangle DAC\| \\ &= \|ABCD\| - \frac{1}{4}\left(\|\triangle ABD\| + \|\triangle CDB\|\right) - \frac{1}{4}\left(\|\triangle BCA\| + \|\triangle DAC\|\right)\end{aligned}$$

In the last line I did a bit of rearranging to match certain triangles together, because they combine to form the whole quadrilateral. Notice that $\|\triangle ABD\| + \|\triangle CDB\| = \|ABCD\|$. Similarly, $\|\triangle BCA\| + \|\triangle DAC\| = \|ABCD\|$. Thus,

$$\|EFGH\| = \|ABCD\| - \frac{1}{4}\|ABCD\| - \frac{1}{4}\|ABCD\| = \frac{1}{2}\|ABCD\|.$$

□

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