This was a tricky exam for all concerned, including me, because the book includes a mixture of "intuitively" recognizing that something is true, and proving it rigorously. In my grading scheme and gradlines my intention was that if you had all of the correct answers and could explain them intuitively, it would be worth about a B+. To get an A you'd have to include further details. However, if I took a few points off of every problem for "lack of rigor," it would quickly bring somebody's score below 80%. Hence I knew I would have to curve the scores, which is one reason I made the test worth 80 points instead of the standard 100; I didn't want people interpreting their scores as a standard percentage. In the end, the scores broke up into the two groups I was expecting. The rough gradelines are: A range, 65-80, and B range, 45-65.

In terms of problem selection, #1 and #2(ii,iii) are essentially examples/definitions from class and the book. Problems #3 and #4 are not specifically examples from the book, but are very similar to examples done in class. The concepts in all of those problems have also come up on many homework problems. #2(i) was a little bit trickier. The intent with that problem was to see if you could combine what we know about open sets, closed sets, and continuous functions to prove a nice fundamental fact.
The solutions below aren't completely comprehensive, but should be enough for you to understand each problem. Let me know if you have any further questions or if you spot any typos.

(1) (a) This is a slight variation of Example 1.7; we did a similar problem in class as well. To construct a bijection $g : [-1,0) \cup (0,1] \to [-1,1]$ I can essentially use the same function as in the example; the "extra" points can just be mapped to themselves:

$$g(x) = \begin{cases} 0, & x = \frac{1}{2} \\ \frac{1}{n-1}, & x = \frac{1}{n}, n = 3,4,5,\dots \\ x, & \text{otherwise.} \end{cases}$$

(b) This is also a slight variant of Examples in the book and from class. To get most of the points, you could point out that $[-1,0) \cup (0,1]$ has two path components, whereas $[-1,1]$ has just one. Since the number of path components is preseved by homeomorphisms, the two spaces can't possible be homeomorphic. To earn full points, you could write a few words of proof showing why each space has the claimed number of path components. (See Example 1.45.)

Note that it's not ok to use the size of each set as a topological invariant, and claim that one set has more points than the other. In the first part of the problem above you've shown they're in bijection, which means they have the same number of points.

(2) (a) Since $C$ is closed, by definition that means $C^c = Y - C$ is open. Because $f$ is continuous, the inverse image of any open set is open, hence $f^{-1}(Y - C) = f^{-1}(C^c)$ is open. But

$$\begin{aligned} f^{-1}(C^c) &= \{x \in X \mid f(x) \in C^c\} \\ &= \{x \in X \mid f(x) \notin C\} \\ &= X - \{x \in X \mid f(x) \in C\} = \left(f^{-1}(C)\right)^c \end{aligned}$$

So really, the first two lines above say that $\left(f^{-1}(C)\right)^c$ is open in $X$, and therefore $f^{-1}(C)$ is closed.

(b) $A \subset X$ is open if and only if for every $a \in A$ there is a number $r > 0$ such that $B_r(a) = \{x \in X \,|\, d(x,a) < r\} \subset A$. (Or, if you prefer, this is Definition 7.7 on page 212.)

(c) This is Theorem 7.9 on page 213, which we also covered twice in class, once for $\mathbb{R}^n$ and once for metric spaces in general.

(3) (a) $\{(x,y) \in \mathbb{R}^2 \,|\, d((x,y),(0,0)) \leq 1\} = \{(x,y) \in \mathbb{R}^2 \,|\max\{|x|,|y|\} \leq 1\}$, so $|x| \leq 1$ and $|y| \leq 1$. This is the square of side length 2 centered at $(0,0)$ described y $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

(b) There are a few different ways to list the conditions to be a metric, and I tried to be generous by accepting as many as possible. Here I'll follow the conditions on page 210 of the textbook.

(i) $d\left((x_1,y_1),(x_2,y_2)\right) = \max\{|x_2 - x_1|,\, |y_2 - y_1|\} \geq 0$ because absolute values are always non-negative. Furthermore, $d\left((x_1,y_1),(x_2,y_2)\right) = 0$ if and only if both $|x_2 - x_1| = 0$ and $|y_2 - y_1| = 0$. which immediately implies that $(x_1,y_1) = (x_2,y_2)$.

(ii) Symmetry follows from the fact that $|a - b| = |b - a|$.

(iii) The triangle inequality is the trickiest of the required properties to prove. We need to show that:

$$d\left((x_1,y_1),(x_3,y_3)\right) \leq d\left((x_1,y_1),(x_2,y_2)\right) + d\left((x_2,y_2),(x_3,y_3)\right)$$

I saw a number of clever approaches to this. The solution I'll present here is less elegant but might be easier to follow for some people. First, consider the left hand side:

$$d\left((x_1,y_1),(x_3,y_3)\right) = \max\{|x_3 - x_1|,\, |y_3 - y_1|\}$$

The right hand side is equal to either $|x_3 - x_1|$ or $|y_3 - y_1|$, depending on which is larger. Suppose it's $|x_3 - x_1|$. Then using the regular triangle inequality, we have

$$|x_3 - x_1| \leq |x_3 - x_2| + |x_2 - x_1|$$

But the right hand side of this last inequality is certainly less than or equal to

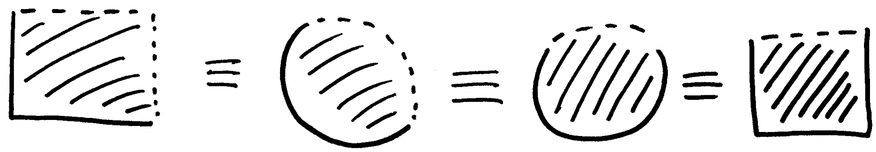
$$\max\{|x_3 - x_2|, |y - 3 - y_2|\} + \max\{|x_2 - x_1|, |y - 2 - y_2|\}$$

(because the first term in that sum is at least as big as $|x_3 - x_2|$, but it might be better, etc.). Hence in this case,

$$\begin{aligned} d\left((x_1,y_1),(x_3,y_3)\right) &\leq \max\{|x_3 - x_2|, |y_3 - y_2|\} + \max\{|x_2 - x_1|, |y_2 - y_1|\} \\ &= d\left((x_1,y_1),(x_2,y_2)\right) + d\left((x_2,y_2),(x_3,y_3)\right) \end{aligned}$$

Going back to our first assumption, suppose $d\left((x_1,y_1),(x_3,y_3)\right)$ is actually equal to $|y_3 - y_1|$. Then we can do similar reasoning which results in the same string of inequalities. In either case the triangle inequality holds.

(4) (a) I wasn't expecting anybody to supply the definition of ambient isotopy here. A series of pictures showing how to continuously deform a square including two of the edges into a square with three of the edges was sufficient. The key point is that every intermediate stage needs to be homeomorphic to the original square. Here's a rough sketch of one approach:



(b) There are three conditions to check, all of which follow from the properties of $=$.

**Reflexive:** It's certainly true that $Df = Df$. (i.e. in calculus notation, $f'(x) = f'(x)$.)

**Symmetric:** Again, if $Df = Dg$, it's certainly true that $Dg = Df$.

**Transitive:** If $Df = Dg$ and $Dg = Dh$, then $Df = Dh$.

The equivalence class of a function $f$ is the set of all functions whose derivatives are equal to $Df$. In other words, it's a family of antiderivatives $f(x) + C$.