These solutions aren't intended to be completely comprehensive, but should at least give you an idea of how to approach each problem. On Homework 1, the eqauivlance class problems in section 1.1 seemed to cause the greatest difficulty, along with 1.2 #23.

Some of these problems are best explained with a blackboard or a piece of paper, so if you have any trouble deciphering the solutions, stop by my office and ask. Also, let me know if you spot any typos and I'll update them as soon as I can.

1.1 #2: **(a):** If $C_a \cap C_b \neq \emptyset$, then there exists an element x which is in both C_a and C_b. By definition, that means $x \sim a$ and $x \sim b$. Since $\sim$ is an equivalence relation, it is both symmetric and transitive, so $a \sim x \sim b$ and $a \sim b$, and it quickly follows that $C_a = C_b$. (Ask me how to show this part that "quickly follows," if you're not sure.)

(b): This is simpler than the authors probably intended. $a \in C_a$, which we could also write as $\{a\} \subset C_a$. Hence

$$\bigcup_{a \in X} \{a\} \subset \bigcup_{a \in X} C_a \subset X$$

The set on the left is just the union of every element of X, so this is clearly X. That means the set in the middle is X as well. (This is analagous to $5 \leq n \leq n$, which forces $n = 5$.)

(c): Let $x \sim y$ iff x and y are both contained in one of the X_α's.

(d): $\sim$ is reflexive (because the union of the X_α's is all of X, x must be contained in one of them, and then x and x are both in the same X_α); symmetric (if x and y are in a set, the order doesn't matter; we could just as well say y and x are in the set); and transitive (if x and y are in X_α, and y and z are in X_β, then in fact we must have $X_\alpha = X_\beta$, and x and z are both contained in it. If not, then $y \in X_\alpha \cap X_\beta = \emptyset$, which is impossible).

1.1 #5: . $a \leq b$ is not symmetric, because (for example) then $1 \leq 2$ would imply that $2 \leq 1$. $a \sim b$ is not reflexive, because if $a = 0$, $aa = 0$ and hence a is not related to itself. $a \approx b$ is not trasitive; suppose $a = 0$, $b = 1$ and $c = 1$. Then $a \approx b$ and $b \approx c$, since $|a - b| \leq 1$ and $|b - c| \leq 1$, but a and c are not related—we have $|a - c| = 2$.

1.1 #10: A sequence $\{a_n\}$ is eventually equal to itself; let $N = 1$. So the relation is reflexive. It's also symmetric: if $a_n = b_n$ for all $n \geq N$, it's certainly true that $b_n = c_n$ for those same values of n. And if

$$\begin{aligned} a_n &= b_n \qquad \text{for all } n \geq N \\ b_n &= c_n \qquad \text{for all } n \geq M \end{aligned}$$

then $a_n = c_n$ for all $n \geq K = \max\{N, M\}$, so the relation is transitive as well.

Note that this is a fairly restrictive notion of when two sequences are eventually equal. You can ask me about a more general notion.

1.2 #2: One possibility would send x to x except: 1/4 to 0 and 3/4 to 1; 1/8 to 1/4 and 7/8 to 3/4; 1/16 to 1/8 and 15/16 to 7/8; and so on.

1.2 #4: **(a):** Suppose f is not an injection. Then there exist $x_1, x_2 \in X$, $x_1 \neq x_2$, such that $f(x_1) = f(x_2)$. But then $g(f(x_1)) = g(f(x_2))$, contradicting the fact that $g \circ f$ is an injection.

An example would be

$$f : [0,1] \to \mathbb{R}, \qquad f(x) = x + 1$$
$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = x^2$$

(b): Suppose g is not an surjection. Then there exists a $z \in Z$ such that $g(y) \neq z$ for all $y \in Y$. But $f(x) \in Y$ for all $x \in X$, so it is impossible to have $g(f(x)) = z$, contradicting the fact that $g \circ f$ is a surjection. An example would be

$$f : \mathbb{N} \to \mathbb{Z}, \qquad f(n) = 2n$$
$$g : \{2, 4, 6, \ldots\} \to \{2, 4, 6, \ldots\}, \qquad g(n) = n$$

(c): If $g \circ f$ is a bijection, then it is surjective and injective. By the first two parts of the problem, f must be an injection, and g must be a surjection. By the two examples, *it is not necessary that g and f be bijections.*

1.2 #7: See Example 1.35 on page 22.

1.2 #22: Prove: for any function $f : X \to Y$, we have $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_Y \circ f = f$.

The proof is an exercise in writing out the defintions. $\mathrm{id}_X(x) = x$ for all $x \in X$, so

$$f \circ \mathrm{id}_X(x) = f(\mathrm{id}_X(x)) = f(x) \text{ for all } x \in X.$$

Similarly,

$$\mathrm{id}_Y \circ f(x) = \mathrm{id}_X(f(x))) = f(x) \text{ for all } y \in Y.$$

1.2 #23: **(a):** First assume that f is a surjection. For any point $y \in Y$, there is at least one "preimage" of y—i.e. a point in X whose value under the function is equal to y. In fact, a given $y \in Y$ might have *many* preimages, but the important thing is that it has at least one. That fact lets us construct the required function g as follows:

$$g(y) = \text{any } x \in X \text{ such that } f(x) = y$$

By definition, $f(g(y)) = y$, so $f \circ g = \mathrm{id}_Y$.

Conversely, suppose such a function exists. Then $f \circ g$ is a surjection from Y to Y (because id_Y certainly is surjective), so I could cite problem 1.2 #4(b) (or else repeat the work from there) to prove that f is a surjection.

(b): Suppose f is an injection. Then any y in the range of f has precisely one preimage, using the terminology from part (a). So we can define the required function g as follows:

$$g(y) = f^{-1}(y) = \text{ the unique } x \in X \text{ such that } f(x) = y$$

By definition, $g(f(x)) = x$, so $g \circ f = \mathrm{id}_X$.

Conversely, suppose such a function exists. Then $g \circ f$ is injective, because $\mathrm{id}_X is$, so I could cite problem 1.2 #4(a) (or else repeat the work from there) to prove that f is injective.

In the extreme case that $X = \emptyset$, this reasoning breaks down because *any* $f : \emptyset \to Y$ is injective—the condition you have to check for any $x_1, x_2 \in \emptyset$ never has to be checked—regardless of what the function g might be.

(c): We'll assume that our sets are non-empty to avoid the weird case in (b); if $X = Y = \emptyset$ then this whole statement is trivial, anyway.

Suppose $f : X \to Y$ is a bijection, so it's both surjective and injective. According to Definition 1.16, g is an inverse function of f if and only if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Parts (a) and (b) of this problem provide functions g which satisfy these equations—**BUT YOU MUST SHOW THAT THEY ARE THE SAME FUNCTION**. In this case the function g provided in (a) really is the same as the g we construct in (b), because f is a bijection, and hence each $f(x)$ has one (and only one) preimage in X.

(d): g and h are equal if they share the same domain and $g(y) = h(y)$ for all points in that domain. In our case, they both have the domain (Y), so we just need to check that the function values agree.

Let y be any element of Y, and consider the function values $g(y)$ and $h(y)$. Using Definition 1.16, we have:

$$f(g(y)) = f \circ g(y) = \mathrm{id}_Y(y) = y$$
$$f(h(y)) = f \circ h(y) = \mathrm{id}_Y(y) = y$$

By part (c) f is a bijection, so in particular it is injective. Then these previous two lines imply that $g(y) = h(y)$, and we're done.

(A): Call the set of irrationals $\mathbb{I}$. Suppose they were countable. Then $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$ would be the union of two countable sets and hence countable. But we showed that $\mathbb{R}$ is uncountable. Hence $\mathbb{I}$ must be uncountable as well.