

These solutions aren't intended to be completely comprehensive, but should at least give you an idea of how to approach each problem. Some of the problems are best explained with a blackboard or a piece of paper, so if you have any trouble deciphering the solutions, stop by my office and ask. Also, let me know if you spot any typos and I'll update them as soon as I can.

1.6 #7: This is one of those problems best explained in person. When I look at the picture the first idea that comes to mind is to shrink the sphere down and "push" it through the loops, untying the knot. However, we're told to keep both spheres fixed! The solution, therefore, is to stretch out the *string*; once the loops are large enough you can simply pull them over the sphere as needed until the knot is untied, and then shrink the string back to a straight line segment.

1.6 #10: These are homeomorphic spaces, so in that topological sense they are the same. However, there is no ambient isotopy which can move the whisker to an ingrown whisker; at some moment in time some part of the whisker (other than the endpoint) would have to intersect the sphere, resulting in a space which is not homeomorphic to the original space. (That's not allowed, because each step of an ambient isotopy must be a homeomorphism.) In \mathbb{R}^4 we can avoid this problem, because the whisker can be "nudged" in the fourth dimension to avoid intersecting the sphere. (If you have trouble understanding that, I can explain it in person.)

7.1 #1(c): Let $P, Q, R \in \mathbb{R}^2$. As the problem indicates, we have to break the proof of the triangle inequality into several cases. Recall that the Roman Road metric d_{RR} is equivalent to the standard Euclidean metric d if two points are on the same line through the origin. I'll break it up into four cases here, although some clever analysis might help you reduce this number.

Case 1: If P, Q and R are all on the same line through the origin, then we just have the triangle inequality with d , which we may assume is true here.

$$d_{RR} = d(P, R) \leq d(P, Q) + d(Q, R) = d_{RR}(P, Q) + d_{RR}(Q, R)$$

Case 2: If P, Q and R are all on *different* rays from the origin, then we can prove the triangle inequality by using the definition of d_{RR} and the fact that $d(O, Q) = d(Q, O) > 0$:

$$\begin{aligned} d_{RR}(P, R) &= d(P, O) + d(O, R) \\ &\leq (d(P, O) + d(O, Q)) + (d(Q, O) + d(O, R)) \\ &= d_{RR}(P, Q) + d_{RR}(Q, R) \end{aligned}$$

Case 3: If P and R are on the same line through the origin, but Q is not, then we can make use of the fact that $d(P, Q) \leq d_{RR}(P, Q)$ and $d(Q, R) \leq d_{RR}(Q, R)$.

$$\begin{aligned} d_{RR}(P, R) &= d(P, R) \leq d(P, Q) + d(Q, R) \\ &\leq d_{RR}(P, Q) + d_{RR}(Q, R) \end{aligned}$$

Case 4: Q is on the same line through the origin as exactly one other point (assume P , so R is not on that line). Here use the fact that $d(P, O) \leq d(P, Q) + d(Q, O)$:

$$\begin{aligned} d_{RR}(P, R) &= d(P, O) + d(O, R) \\ &\leq (d(P, Q) + d(Q, O)) + d(O, R) \\ &\leq d(P, Q) + (d(Q, O) + d(O, R)) \\ &= d_{RR}(P, Q) + d_{RR}(Q, R) \end{aligned}$$

7.1 #9: (a): Let $B_r(a, b)$ and $B_r^{TC}(a, b)$ represent the open ball of radius r centered at (a, b) in the standard and Taxi Cab metrics, respectively. You can show quickly with a picture that $B_r^{TC}(a, b) \subset B_r(a, b)$. It comes down to the fact that a square with diagonal length $2r$ can be inscribed in a circle of radius r . I can think of two ways to prove it:

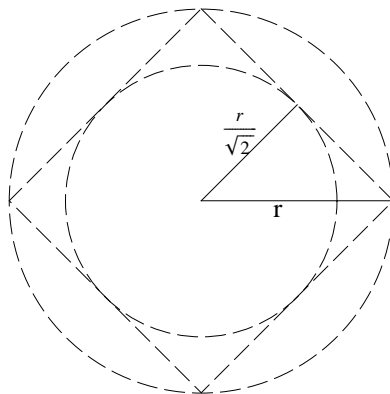
(1) Let $(x, y) \in B_r^{TC}(a, b) = \{(x, y) : |x - a| + |y - b| < r\}$. Then

$$\begin{aligned} |x - a| + |y - b| &< r \\ (x - a)^2 + 2 \cdot |x - a| \cdot |y - b| + (y - b)^2 &< r^2 \\ (x - 2)^2 + (y - b)^2 &< r^2 - 2 \cdot |x - a| \cdot |y - b| < r^2 \end{aligned}$$

Hence $(x, y) \in B_r(a, b) = \{(x, y) : (x - a)^2 + (y - b)^2 < r^2\}$

(2) Another possibility is to show that, for any two points P and Q in the Euclidean plane, $d(P, Q) \leq d_{RR}(P, Q)$. Just looking at balls centered at the origin (for simplicity of notation), this would imply that any point P for which $d_{RR}(P, O) < r$ certainly has $d(P, O) < r$ as well. (This is because the Roman Road metric gives you distances which are at least as big as the standard metric.)

(b): Again, you can show with a picture – and prove algebraically – that $B_{r/\sqrt{2}}(x, y) \subset B_r^{TC}(x, y)$. Both (a) and (b) are illustrated in the following picture:



(c): Remember the definition of open set: every point is an interior point. So imagine that U is an open set with the Euclidean metric d . That means for any point $(x, y) \in U$, there is a radius r so that $B_r(x, y) \subset U$. But by part (a), we also have $B_r^{TC}(x, y) \subset B_r(x, y) \subset U$, so that (x, y) is an interior point when drawing balls with the Taxi Cab metric, too. That means U is an open set with the Taxi Cab metric..

The same argument works in the other direction; if U is open with the Taxi Cab metric, that means for any $(x, y) \in U$, there is a radius r so that $B_r^{TC}(x, y) \subset U$. But then $B_{r/\sqrt{2}}(x, y) \subset B_r^{TC}(x, y) \subset U$, so U is also open with the Euclidean metric.

We've now shown that the open sets are exactly the same with these two metrics.

7.1 #11: Let U be any subset of \mathbb{R}^2 , and take any $x \in U$. Using the discrete metric, $B_{1/2}(x) = \{x\} \subset U$, so every point in U is an interior point. Hence U is open.

7.1 #12: (a): Let $x \in A \cap B$. Then $x \in A$, so $f(x) \in f(A)$. Similarly, $f(x) \in f(B)$. Hence $f(x) \in f(A) \cap f(B)$. Since this was true for any $x \in A \cap B$, we have $f(A \cap B) \subset f(A) \cap f(B)$.

(b): One possibility is $f(x) = x^2$, $A = [-1, 0]$, $B = [0, 1]$. Then $A \cap B = \{0\}$ and $f(A \cap B) = \{0\}$. But $f(A) \cap f(B) = [0, 1] \cap [0, 1] = [0, 1]$.

(c): Let $y \in C \cap D$, so y is in both C and D . Then $f^{-1}(y) \in f^{-1}(C)$ and $f^{-1}(y) \in f^{-1}(D)$, so $y \in f^{-1}(C) \cap f^{-1}(D)$. Thus $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$.

Conversely, suppose $x \in f^{-1}(C) \cap f^{-1}(D)$, so x is a preimage of points in both C and D . Put differently, $f(x)$ is in both C and D , or $f(x) \in C \cap D$. But then $x \in f^{-1}(C \cap D)$. Hence $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$.

Having shown both directions, we can say that $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D)$, and this is true for arbitrary unions, not just finite ones.

(d): The situation is somewhat reversed with unions; I'll leave the details to you, but you can ask me if you have any questions. In particular, we have $f^{-1}(C) \cup f^{-1}(D) = f^{-1}(C \cup D)$.

K1: Although a $\epsilon - \delta$ proof is possible, I think it's easier to use our third characterization of continuous function in this case. Let $(a, b) \subset \mathbb{R}$ be any open interval, and consider $f^{-1}((a, b))$:

$$f^{-1}((a, b)) = \{x \in X \mid a < d(x_0, x) < b\}$$

This is an open annulus in X . (Think of the set $\{1 < x^2 + y^2 < 4\}$ in Euclidean space.) Hence the inverse image of any open interval in \mathbb{R} is an open set in X .

Now we use the fact that *any* open set U in \mathbb{R} is a union of intervals I_α , and the result of 7.1 #12(d), to show that the inverse image of any open set is open:

$$f^{-1}\left(\bigcup_{\alpha} I_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(I_{\alpha})$$

The right hand side is a union of open sets, hence is open itself, and we're done.