UMTYMP Calc III Fall 2005 Exam 3 Review Solutions

These are mostly complete solutions to the review sheet we used in class. Don't be thrown off by the changes in notation here and there (i.e. u and v versus θ and ϕ).

Please remember that there are a few topics on the exam which aren't stressed in the review problems – surface integrals of *scalar* functions, for example. Also let us know if you need help with the extra problem, computing the volume of the solid in the first octant bounded by $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

Earlier versions of these solutions had a **typo** in 4(b). Check your work against the current version and let us know if you have any questions.

1. As the given region is a cylinder, and the integrand is suitable, it is convenient to express the required integral in cylindrical coordinates:

$$\iiint_B 2 - 2z(x^2 + y^2)\, dx\, dy\, dz = \int_0^{2\pi} \int_0^3 \int_0^2 \left(2 - 2zr^2\right) \cdot r\, dz\, dr\, d\theta$$
$$= -126\pi$$

2. The surface S, which is the boundary of the solid region B of Question 1, consists of the bottom, lateral, and top faces of the cylinder, which we shall denote S_1, S_2, and S_3 respectively:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

Bottom Face:
The bottom face S_1 is the disk of radius 3 centered at the origin and lying in the xy-plane. It can be parametrized as:

$$\mathbf{f}(r, \theta) = (r\cos(\theta), r\sin(\theta), 0) \text{ with } 0 \le r \le 3 \text{ and } 0 \le \theta \le 2\pi$$

We easily compute that:

$$\frac{\partial \mathbf{f}}{\partial \theta} \times \frac{\partial \mathbf{f}}{\partial r} = (0, 0, -r)$$

so that:

$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \int_0^{2\pi}\int_0^3 \mathbf{F}\left(\mathbf{f}(r,\theta)\right)\cdot\left(\frac{\partial\mathbf{f}}{\partial\theta}\times\frac{\partial\mathbf{f}}{\partial r}\right)\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^3 (0,0,4r^2)\cdot(0,0,-r)\,dr\,d\theta \\ &= -\int_0^{2\pi}\int_0^3 4r^3\,dr\,d\theta = -162\pi\end{aligned}$$

Lateral Side:
Lateral (vertical) side of the cylinder can be parametrized, for instance, by:

$$\mathbf{f}(\theta,z) = (3\cos(\theta), 3\sin(\theta), z)$$

where

$$0 \le \theta \le 2\pi \text{ and } 0 \le z \le 2\pi$$

We easily compute:

$$\begin{aligned}\frac{\partial\mathbf{f}}{\partial\theta}\times\frac{\partial\mathbf{f}}{\partial z} &= (3\cos(\theta), 3\sin(\theta), 0) \\ \mathbf{F}\left(\mathbf{f}(\theta,z)\right) &= \left(3\cos(\theta), 3\sin(\theta), 9(4-z^2)\right)\end{aligned}$$

so that:

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \int_0^2\int_0^{2\pi} \left(3\cos(\theta), 3\sin(\theta), 9(4-z^2)\right)\cdot(3\cos(\theta), 3\sin(\theta), 0)\;d\theta\,dz \\ &= \int_0^2\int_0^{2\pi} 9\,d\theta\,dz \\ &= 36\pi\end{aligned}$$

Top Side: The integral over the top side is 0; this is seen without any real effort by simply noting that over this part of the boundary the normal points in the vertical direction (i.e. (0,0,1)) while the z-component of the vector field $\mathbf{F}$ is 0 when $z = 2$.

Putting it all together, we find:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = -162\pi + 36\pi + 0 = -126\pi$$

3. They are equal; this is an instance of the Divergence (Gauss') Theorem.

4. *Part (a)*
   The hemisphere and paraboloid intersect in a circle of radius $2\sqrt{3}$ in the plane $z = 2$. The solid can be described as follows in cylindrical coordinates:

$$\begin{aligned} 0 \leq & r \leq 2\sqrt{3} \\ 0 \leq & \theta \leq 2\pi \\ \frac{r^2}{6} \leq & z \leq \sqrt{16 - r^2} \end{aligned}$$

   Part (b):
   Div $\mathbf{F}(x, y, z) = 1 + 0 + 0 = 1$, so the Divergence Theorem says that the flux is just the integral of 1 over the solid.

$$\iiint_S 1\, dV = \int_0^{2\pi} \int_0^{2\sqrt{3}} \int_{r^2/6}^{\sqrt{16-r^2}} r\, dz\, dr\, d\theta = \frac{76}{3}\pi$$

5. *Part (a):* There are many possibilities. One way to parametrize the whole surface is:

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^2 - 9 \rangle, \qquad 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$$

   One the boundary, $u = 3$, so the boundary is the circle of radius 3 in the xy-plane. However, we want the *positively* oriented boundary, which means the direction needs to be consistent with the outward (or downward, in this case) pointing normal vector. That means we need to traverse the circle *counterclockwise as seen from* ***below***.

$$\mathbf{r}(t) = \langle 3 \sin t, 3 \cos t, 0 \rangle, \qquad 0 \leq t \leq 2\pi$$

   Part (b): One look at the formula for $\mathbf{F}$ should be enough to convince you never to compute its curl. Instead we use Stokes Theorem, which says that

$$\begin{aligned}\iint_M \text{curl } \mathbf{F} \cdot d\mathbf{S} &= \oint_{\partial M} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} \langle -3\cos t, 3\sin t, 0\rangle \cdot \langle 3\cos t, -3\sin t, 0\rangle\, dt \\ &= \cdots = \int_0^{2\pi} -9\, dt = -18\pi\end{aligned}$$

6. We can't use the Divergence Theorem directly here, because the surface is *not* the positively oriented boundary of the upper hemisphere of radius 2. (It's missing the bottom.) Let's compute the flux integral directly. First parametrize the surface using spherical coordinates (where $\rho = 2$).

$$\mathbf{r}(u, v) = \langle 2\cos u \sin v, 2\sin u \sin v, 2\cos v\rangle, \qquad 0 \le u \le 2\pi,\ 0 \le v \le \pi/2.$$

You can calculate that the outward pointing normal vector is $\mathbf{r}_v \times \mathbf{r}_u$. Then

$$\begin{aligned}\iint_M \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_v \times \mathbf{r}_u)\, dudv &= \int_0^{2\pi} \int_0^{\pi/2} (\text{a big mess....})\, dvdu \\ &= \cdots = \frac{16\pi}{3}\end{aligned}$$

7. There is a very fast way to write down the surface integral we are asked for in the first part of this problem, but here is how you'll probably be more comfortable doing it: We use spherical coordinates to parametrize the upper hemi-sphere, like so:

$$\mathbf{f}(\phi, \theta) = (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$$

where $0 \le \phi \le \pi/2$ and $0 \le \theta \le 2\pi$. A straightforward calculation, utilizing the usual trigonometric identity, gives:

$$\begin{aligned}\frac{\partial \mathbf{f}}{\partial \phi} \times \frac{\partial \mathbf{f}}{\partial \theta} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\ &= \sin(\phi)\,(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))\end{aligned}$$

which points in the outward normal direction, as the question requires. We also calculate:

$$\nabla\times\mathbf{F}\left(\mathbf{f}(\phi,\theta)\right) = \big(2\sin(\phi)\cos(\theta)\cos(\phi), 2\sin(\phi)\sin(\theta)\cos(\phi), 2\cos^2(\phi)\big)$$

so that:

$$\nabla\times\mathbf{F}\left(\mathbf{f}(\phi,\theta)\right)\cdot\left(\frac{\partial\mathbf{f}}{\partial\phi}\times\frac{\partial\mathbf{f}}{\partial\theta}\right) = 2\sin(\phi)\cos(\phi)$$

So our integral is:

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 2\sin(\phi)\cos(\phi)\,d\phi\,d\theta$$

Turning now to the second part of the problem, we first quote Stokes' Theorem:

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_{\partial S}\mathbf{F}\cdot d\mathbf{x}$$

In our case, the boundary ∂S of the surface S is the circle of radius 1 centered at the origin, and lying in the xy-plane. The correct orientation for the curve, given that of the surface, is the counter-clockwise one; one such parametrization is:

$$\gamma(\theta) = (\cos(\theta), \sin(\theta), 0) \text{ where } 0\le\theta\le 2\pi$$

We find:

$$\begin{aligned}\int_{\partial S}\mathbf{F}\cdot d\mathbf{x} &= \int_0^{2\pi}\mathbf{F}\left(\gamma(\theta)\right)\cdot\gamma'(t)\,dt\\ &= \int_0^{2\pi}\left(-\sin(\theta),\cos(\theta),0\right)\cdot\left(-\sin(\theta),\cos(\theta),0\right)\,dt\\ &= \int_0^{2\pi} 1\,dt = 2\pi\end{aligned}$$