Note: Some of these review problems are hard, much harder than the actual exam problems. That's intentional! If you can handle these kinds of problems with ease (and without looking things up in your book, etc.) you should be fine on the test.

1. Consider a particular moving in $\mathbb{R}^2$ whose position vector is given by

$$\boldsymbol{r} = \boldsymbol{r}(t) = R\cos(kt)\boldsymbol{i} + R\sin(kt)\boldsymbol{j},$$

where R and k are nonzero constants.

For what values of k...

(a) is $\boldsymbol{r}$ a unit-speed (speed is 1 for all t) parametrization of the circle?
(b) is $\boldsymbol{r}$ a constant-speed parametrization?
(c) is the velocity vector perpendicular to the position vector?
(d) is the velocity vector equal to the unit tangent vector?
(e) is the accleration vector perpendicular to the velocity vector? is the acceleration vector parallel to the unit tangent vector?

Calculate κ. Does it depend on t? Does it depend on the choice of k? How is the value of κ related to the radius of the osculating circle?

2. What can you say about the angle between the velocity and acceleration vectors if the speed is constant?

3. Now consider a parametric curve of the form

$$\boldsymbol{r}(t) = R\cos(f(t))\boldsymbol{i} + R\sin(f(t))\boldsymbol{j},$$

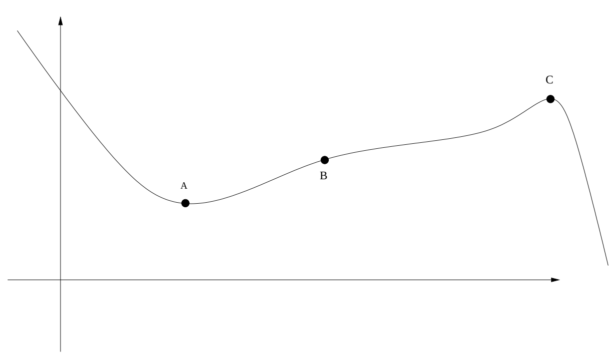
where R is a constant and $f(t)$ is a function of t.

(a) Find a function $f(t)$ so that, for this parametric curve, the acceleration vector is *never* perpendicular to the velocity vector.
(b) Assuming $f(t)$ is the function you found in part (a), calculate v', where $v = v(t)$ is the speed at time t. Draw a sketch to show the vectors $\boldsymbol{r}(t)$, $\boldsymbol{v}(t)$, and $\boldsymbol{a}(t)$ for some time t. Also draw $\boldsymbol{T}(t)$.

4. Find a curve C with parametrization $\boldsymbol{r}(t)$ which has all the following properties:

- the curve lies on a cylinder whose central axis is parallel to the z-axis,
- the projection of the space curve onto the yz-plane is a circle of radius 2 centered at $(6, 4)$,
- the space curve passes through the points $(0, 4, 4)$ and $(\pi, 4, 4)$.

5. Consider the vector function $\boldsymbol{r}(t)$ describing the curve shown below. Put the curvatures of the curve at A, B, and C in order from smallest to largest.



6. Three particles Q_1, Q_2, and Q_3 travel in an endless loop around a closed curve in $\mathbb{R}^3$. Let $\boldsymbol{r}_1(t)$, $\boldsymbol{r}_2(t)$, and $\boldsymbol{r}_3(t)$ be their respective position vectors at time t; assume these are all smooth vector functions. Suppose that $\boldsymbol{r}_1(5) = \boldsymbol{r}_2(5) = \boldsymbol{r}_3(5)$. (That is, they are at the same place when $t = 5$.)

   For each set of quantities below, decide whether they must all be identical, they can have two different values, or they can all be distinct. Note that the particles can be moving around the curve in opposite directions.

   (a) The three speeds $|\boldsymbol{r}_1(5)|$, $|\boldsymbol{r}_2(5)|$, $|\boldsymbol{r}_1(5)|$.

   (b) The three unit tangent vectors $\boldsymbol{T}_1(5)$, $\boldsymbol{T}_2(5)$, $\boldsymbol{T}_3(5)$.

   (c) The three curvatures $\kappa_1(5)$, $\kappa_2(5)$, $\kappa_3(5)$.

   (d) The Three Musketeers Athos(5), Porthos(5), Aramis(5).

7. Find a vector function $\boldsymbol{r}(t) = (f(t), g(t), h(t))$ such that $\boldsymbol{r}(0) = (0, 2, 0)$, $r(1) = (0, 4, 0)$, $\boldsymbol{r}''(t) \neq \boldsymbol{0}$, and $\boldsymbol{r}'(t) \parallel \boldsymbol{r}''(t)$ for all t. Show that your answer is correct.

   Also find a vector function $\boldsymbol{r}(t) = (f(t), g(t), h(t))$ such that $\boldsymbol{r}(0) = (0, 2, 0)$, $r(1) = (0, 4, 0)$, $\boldsymbol{r}''(t) \neq \boldsymbol{0}$, and $\boldsymbol{r}'(t) \perp \boldsymbol{r}''(t)$ for all t. Show that your answer is correct.

8. Sketch the given subset of $\mathbb{R}^3$:

   (a) the subset of $\mathbb{R}^3$ defined in cylindrical coordinates by $r = 3 \sec \theta$

   (b) the subset of $\mathbb{R}^3$ defined in cylindrical coordinates by $\cos \theta$

   (c) the subset of $\mathbb{R}^3$ defined in spherical coordinates by $\rho = \cos\left(\frac{\pi}{2} - \varphi\right)$, $0 \leq \theta \leq \pi$.

9. Consider the point (π, π, π). In which coordinate system—cartesian, spherical, or cylindrical—is this point farthest from the origin?

   You can also read about other coordinate systems at this link: http://urlx.org/wolfram.com/e3915. Can you find a coordinate system in which the point (π, π, π) is closer to the origin than any of our standard three coordinate systems? Farther?

   (No coordinate system other than the main three will appear on the exam, or indeed, in our class.)

10. Simon is trapped on a very small planet and is dying of thirst. He needs to find his coffee mug or he will collapse and never be able to use the decapitation stick again. He is standing on the north pole of the planet, and wants to walk around it in a spiral until he gets to the south pole. If he spirals around the planet 15 times, he will be able to find his coffee mug and quench his thirst. Parametrize a curve for Simon to walk along that will insure his survival. Assume the planet has radius 10. Be sure to specify what coordinate system you are using.

11. Sketch the surface $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$.

12. Find the arclength of the curve described by $\boldsymbol{r}(t) = (\ln t, t^2/2, t\sqrt{2})$ for $1 \leq t \leq 4$. Then set up the integral for the arclength function $s(t)$ for the curve $\boldsymbol{r}(t) = (t, t\sqrt{2}, t^2)$ for nonnegative t. You don't need to evaluate the integral.

13. Simon managed to find his coffee mug. It was at the south pole, so he had to walk around the planet all 15 times. How far did he walk? Here is one possible solution:

    > He went around the planet 15 times. If he had walked around a cylinder of radius 10 that many times, he would have walked $15 \cdot 2\pi r = 300\pi$ units. But his radius started at 0, increased to 10, and then decreased to 0 again. By symmetry, the average radius was 5, so he walked $15 \cdot 2\pi r = 150\pi$ units.

    Is this reasoning correct? If not: (a) can you fix it so that it is, thereby avoiding doing a nasty integral? (b) can you change the shape of the planet so that the above reasoning works?

14. (**Note**: This is a previous test question; it was a great problem, but a pain to grade, so we're putting it on the review this time.) Consider the curve parametrized by

$$\boldsymbol{r}(t) = \langle \frac{t^2}{2}, \frac{t^4}{\sqrt{8}}, \frac{t^6}{6} \rangle, \qquad -\infty < t < \infty$$

(a) Briefly describe (in words) the behavior of the curve near $t = 0$.

(b) Evaluate $\lim_{t\to 0} \boldsymbol{r}'(t)$ and $\lim_{t\to\infty} \boldsymbol{r}'(t)$. If either does not exist, explain why not.

(c) Evaluate $\lim_{t\to 0} \boldsymbol{T}(t)$ and $\lim_{t\to\infty} \boldsymbol{T}(t)$. If either does not exist, explain why not.

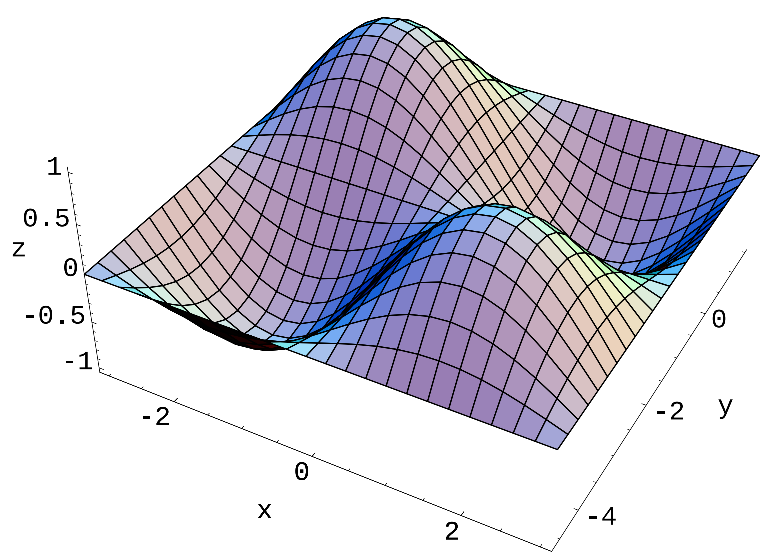
(d) Find $\boldsymbol{T}(1)$ and $\boldsymbol{N}(1)$. You do not have to find a general expression for $\boldsymbol{N}(t)$.

(e) Parametrize the osculating plane of the curve at the point $\boldsymbol{r}(1)$.

15. Sketch and describe the level sets (curves) of $f(x, y) = (x - y)^2$ in the xy-plane.

16. Describe the level sets (surfaces) of the function $g(x, y, z) = z - (x - y)^2$. Sketch the level set $g(x, y, z) = 0$.

17. Sketch the level sets of the following surface:



18. Sketch the "rollercoaster" curve $\boldsymbol{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$, $0 \le t \le 2\pi$. At which points on the curve is the acceleration greatest, and in which direction is it pointing at those places? (Note that your tummy may well feel as if it is moving in the opposite direction to the acceleration.) Can you find the osculating plane when $t = 0$ or $t = \pi/2$?

19. Consider the curve parametrized by

$$\boldsymbol{r}(t) = \begin{cases} \langle t - 2, 1, 0 \rangle, & 0 \le t \le 2 \\ \langle \sin(t-2), \cos(t-2), t - 2 \rangle, & 2 \le t \le 2 + \pi/2 \\ \langle 1, t - 2 + \pi/2, 1 \rangle, & 2 + \pi/2 \le t \end{cases}$$

Where is the acceleration defined in this interval? Where is the acceleration in the direction of motion? Where is it perpendicular to the direction of motion? Sketch the velocity and acceleration vectors at $t = 2 + \pi/4$, placing their tails at the end of the position vector.

Is the acceleration always horizontal when it is defined? Could you change your answer to that question by reparametrizing the curve?

20. Sketch and describe the surfaces in $\boldsymbol{R}^3$ determined by the following equations.

$$\frac{x^2}{4} - \frac{y^2}{16} + \frac{z^2}{9} = 1$$
$$x^2 + \frac{y^2}{9} - \frac{z^2}{16} = 0$$
$$y^2 + 2 = z$$