UMTYMP Calculus III Quiz 2 review problems

1. Consider the curve $C = C_1 \cup C_2 \cup C_3$, where:
   - C_1 goes from the origin to (0,3,0) along a straight line segment.
   - C_2 goes from (0,3,0) to (0,3,-5) along a straight line segment.
   - C_3 goes from (0,3,-5) to (2,3,-5) along a straight line segment.

a) Sketch C_1, C_2 and C_3 in $\mathbb{R}^3$.
b) Without doing any computations other than addition and multiplication of real numbers, evaluate $\int_C yz\,dx + x\,dy + y\,dz$.

2. Explain geometrically why each of the following line integrals evaluates to zero:
a) $\int_C e^{\arctan(x^4)} y^3 \cos(2y)\; ds$, where C goes from (10,15) to (10,-15) along a straight line segment.
b) $\oint_C \frac{x}{x^2+y^3+1}dx + \frac{y}{x^2+y^3+1}dy$, where C is the unit circle oriented counterclockwise.
c) $\int_C -e^{xy} y\, dx + e^{xy} x\, dy$, where C is any line segment that lies on a line passing through the origin.

3. Identify the following sets as connected or disconnected, and as open or not open.

a) $S = \{(x,y) \mid x \ge 0,\ y \ge 0,\ x+y \le 1\}$
b) $T = \{(x,y) \mid x < 0 \text{ or } x > 3\}$
c) $K = \{(x,y) \mid |x| \ge 1 \text{ or } |y| \ge 1\}$
d) $U = \{(x,y) \mid x^2+y^2 < 1 \text{ or } (x-3)^2 + y^2 < 4\}$
e) $B = \{(x,y) \mid x^2+y^2 < 81\} \cup \{(x,y) \mid x^2+y^2 > 81\}$
f) $B = \{(x,y) \mid 0 < (x-3)^2 + (y+1)^2 < 16\}$

4. Which of the connected sets in Exercise 3 are simply-connected?

5. Let $\mathbf{F} = P(x)\,\mathbf{i} + Q(y)\,\mathbf{j}$ be a vector field in an open simply-connected region D, and P and Q have continuous first-order partial derivatives. Is $\mathbf{F}$ conservative? If so, find f such that $\mathbf{F} = \nabla f$.

6. Compute $\oint_C \left((\cos x + 1)^2 + 2y \right) dx + \left((\sin y - 1)^2 - 5x\left(1 + \frac{1}{x}\right) \right) dy$, where C is consists of the line segments from (1, 0) to (0, 1), from (0, 1) to (-1, 1), from (-1, 1) to (-1, 0), and the lower half of the unit circle from (-1, 0) back to (1, 0).

7. If R is the shaded region consisting of the two disks shown below, whose boundary ∂R is oriented as shown, evaluate $\int_{\partial R} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \left(2x^2 - 3y\right)\mathbf{i} + \left(5x + y^{10}\right)\mathbf{i}$.

8. Let R be the top half of the ellipse $x^2 + \frac{y^2}{4} = 1$.

a) Find a counterclockwise parameterization of the boundary ∂R of R

b) Compute the double integral $\iint_R 3x^2 y\, dA$. Hint: Can you find a vector function $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 y$?

9. Let C be the line segment from (a, b) to (c, d), where $a < c$ and $b > d$. Compute $I_1 = \int_C e^{-x} dx$, $I_2 = \int_C e^{-x} dy$ and $I_3 = \int_C e^{-x} ds$. Explain (by thinking about what the integrals actually represent) the signs of I_1, I_2 and I_3. Also explain why I_1 doesn't depend on b and d.