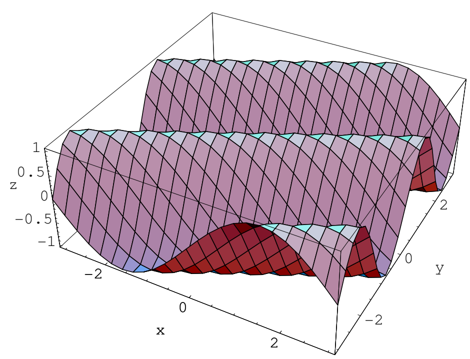
This writeup should give you an idea of what I would look for when grading this problem. It is is *not* intended to be a thorough solution.

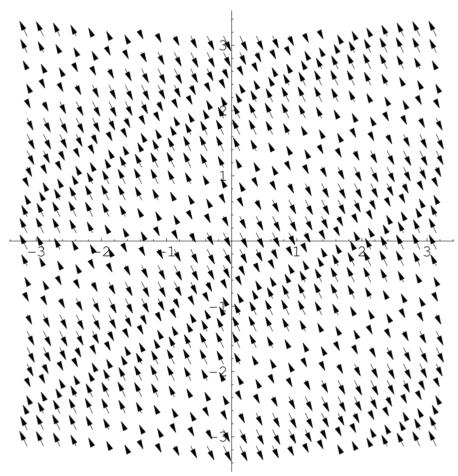
#26. Let $f(x,y) = \sin(x-2y)$ and let $F = \bigtriangledown f$. Find two curves C_1 and C_2 which are not closed and satisfy

$$\text{(a)} \int_{C_1} F \cdot d\vec{r} = 0$$

$$\text{(a)} \int_{C_1} F \cdot d\vec{r} = 1$$

A graph of the function $f(x,y$ and its gradient field F are shown here.





Of course, it's possible to do this problem without ever looking at a picture, but in a professional problem you're trying to provide a lot of detail, and pictures can be useful. In particular, you could use the picture of the vector field to help confirm your answers; more on that below.

If you don't have a good feel for what this vector field is doing, recall that the gradient points in the direction of steepest increase on a surface. If you look at the pictures above, you should be able to convince yourself that the vectors are pointing directly toward the peaks on the surface, and away from the troughs.

By definition, F is a conservative vector field, so we can use the Fundamental Theorem of Line Integrals, which says that for any curve C which starts at $\vec{a}$ and ends at $\vec{b}$,

$$\int_C F \cdot d\vec{r} = \int_C \bigtriangledown f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

Thus we want to choose points $\vec{b}$ and $\vec{a}$ such that $f(\vec{b}) - f(\vec{a}) = 0$ or 1.

Here's the point I want to make: nearly everybody made this problem much more difficult than it needed to be. Most people set up systems of linear equations to find the x- and y-values of the endpoints. All of this was unnecessary.

To make life easier, let's just arbitrarily assume that $\vec{a} = (0,0)$. Since $f(0,0) = 0$, this reduces our problem to choosing $\vec{b}$ such that $f(\vec{b}) = 0$ or 1.

For part (a), just choose a point $(x,y) \neq (0,0)$ such that $\sin(x-2y) = 0$. Again, make life easy: choose $y = 0$, in which case $x = \pi$ works just fine, since $\sin(\pi - 0) = 0$. Thus C_1 is a curve from $(0,0)$ to $(\pi, 0)$. The simplest such curve is probably the line segment between the two points. Then

$$\int_{C_1} F \cdot d\vec{r} = f(\pi, 0) - f(0,0) = 0 - 0 = 0$$

For (b), let's continue to assume $y = 0$. Thus we need a value of x such that $\sin(x) = 1$. Well, $x = \frac{\pi}{2}$ works just fine, so we can choose C_2 to be a curve from $(0,0)$ to the point $(\pi/2)$. Again, the line segment between these two works just fine. We have

$$\int_{C_2} F \cdot d\vec{r} = f(\pi/2, 0) - f(0,0) = 1 - 0 = 1$$

Many people wrote these equations as a "confirmation" of their work. But this doesn't really confirm it; it's just a restatement of the earlier work. In other words, we chose C_1 and C_2 specifically so that their endpoints satisfied these equations; writing the equations down again doesn't really add anything to our knowledge.

What should you have done? Well, you could check your answers by parametrizing the curves and computing the line integrals according to the definition. Alternatively, you could have used pictures to visually estimate the integrals. A closeup of the vector field together with the two curves is shown below. In the first case, it's very plausible that the answer is 0; as you move along the curve, the vectors in the vector field cancel out. In the second picture, you'd be hard pressed to visually estimate that the answer is exactly 1, but you can at least tell that the answer is positive, which would have been good enough for the confirmation point.