The following solutions are not very complete, but they should in general give you a good idea of how to solve the problem. These solutions haven't been carefully checked, so there may be errors. I've omitted the book problems.

**Problem 1.** (a) The gradient vector is perpendicular to the level surface, and hence is a normal vector for the tangent plane. The gradient is $(3x^2y, x^3 - z^2, 2yz + 5z^4)$. At $(3, -1, 2)$ the gradient is $(-27, 23, 76)$ and the equation of the tangent plane is

$$-27(x - 3) + 23(y + 1) + 76(z - 2) = 0.$$

(b) If the tangent plane is parallel to $x + y + z = 1$, it must have a normal vector parallel to $(1, 1, 1)$. The gradient of our level set is parallel to $(1, 1, 1)$ when

$$(4x^3, 4y^3, 4z^3) = c(1, 1, 1)$$

for some nonzero value of $c$. We just need $x$, $y$, and $z$ to be equal; you can check that such points on the curve are $\pm(1/\sqrt[3]{3}, 1/\sqrt[3]{3}, 1/\sqrt[3]{3})$.

(c) If the two surfaces are to be tangent, their gradient vectors must be parallel. We use ideas just like in part (b): we need to find points where

$$(x/2, 2y, 2z) = k(2x, 2y, -2(z + 1)).$$

The second components of those vectors force $k = 1$; then we must have

$$x/2 = 2x \quad \text{and} \quad 2z = -2z - 2,$$

which clearly makes $x = z = 0$. Then $y = \pm 1$ in order to be on the ellipsoid; you can figure out exactly what value of $c$ gives the proper hyperboloid.

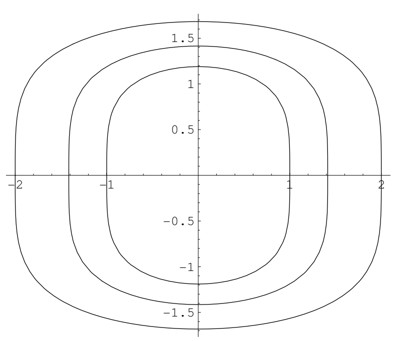
**Problem 2.** This problem should read "...points on the curve $\sin(x) + \cos(y) = 0$ *where the tangent line* is parallel to the line $y = x$."

Again, we use the gradient vector, which will be perpendicular to the direction vector of the tangent line. The gradient is $(\cos x, -\sin y)$, and we need to find points where that's perpendicular to $(1, 1)$. That will happen when the components of the gradient have opposite sign, or when $\cos x = \sin y$.

**Problem 3.** Part (a) was rather difficult. One way to do it is to integrate: note that the function $f(x, y, z) = x^2/2 + y^4/4 + z^2/2$ has the given gradient. If $z = 1$, we are finding level sets of the function

$$x^2/2 + y^4/4 = c,$$

where $c$ is some constant. These sets will look like boxy ellipses. Here are some of the level sets:




Part (b) is much more straightforward. Find when the gradient is parallel to the normal vector of the plane. You'll get $x = z = 1$ and $y = \sqrt[3]{1/4}$.

**Problem 4.** The gradient is $(2nx^{2n-1}, 2ny^{2n-1})$. We need to find where that vector is perpendicular to the direction vector of the line, which is $(1, -2)$. Therefore we need

$$(2nx^{2n-1}, 2ny^{2n-1}) \cdot (1, -2) = 0,$$

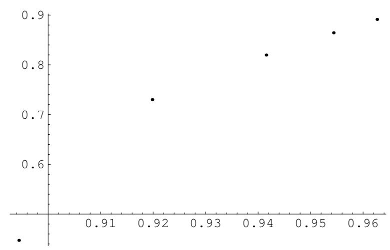
which means that

$$\begin{aligned} 2nx^{2n-1} &= 4ny^{2n-1} \\ x &= (2y^{2n-1})^{1/(2n-1)} \\ x &= 2^{1/(2n-1)}y. \end{aligned}$$

Now solve for $y$ by using the equation $x^{2n} + y^{2n} = 1$. You'll find that

$$y = \left( \frac{1}{2^{(2n)/(2n-1)} + 1} \right)^{1/2n}.$$

As $n \to \infty$, $x$ and $y$ both go to 1. Here are the first 5 $(x, y)$ pairs:




**Problem 5.** If the tangent plane contains the vector $(0, 1, 1)$ iff the normal vector to the tangent plane is perpendicular to $(0, 1, 1)$. Find all points where the gradient is perpendicular to that vector: you'll need $y = -z$, and you can find out the relation between $x$ and $y$ by using the equation of the sphere: $x^2 + 2y^2 = 1$. All $(x, y, z)$ satisfying these two equations will have the correct tangent plane.

Now in part (b), observe that in the ellipsoid, $x$ can range from $-3$ to 3, $y$ from $-6$ to 6, and $z$ from $-3$ to 3. The simplest linear transformation that does this is given by

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Part (c): the easiest way is to proceed exactly as in part (a), but with the gradient vector $(2x, y/2, 2z)$ and the vector $(0, 2, 1)$. You'll again get two equations that $x$, $y$, and $z$ must satisfy.

How might you use part (b) to solve this equation? Can you use that linear transformation to "transport" the problem from the sphere to the ellipsoid, or vice versa?

**Problem 7.** Sylvester's Theorem fails to tell us anything about this matrix, since the upper-left entry of the matrix is zero. So we need to find points where this quadratic form is positive, and where it is negative.

Begin with $(1, 1, 0)$. Since $p(1, 1, 0) = 2a$, we have a positive or negative point, depending on whether $a$ is positive or negative. On the other hand, $p(-1, 1, 0) = -2a$, which has the opposite sign.

**Problem 9.** This is quite straightforward. Find the value of the function, the gradient and the Hessian, evaluate them at $(0, 0)$, and you get

$$f(0,0) + \nabla f(0,0) \cdot \boldsymbol{x} + \boldsymbol{x}^T Hf(0,0)\boldsymbol{x} = 1 + 0 - x^2/2 - y^2/2 + x^2y^2/4.$$

(Curiously, that's exactly the product of the second-order Taylor polynomials for $\cos x$ and $\cos y$: it's $(1 - x^2/2)(1 - y^2/2)$. I don't know if that's a general fact. You can try to prove this: *If $f(x, y) = g(x)h(y)$, then the second-order Taylor polynomial of $f$ equals the product of the second-order Taylor polynomials for $g$ and $h$.*)

**Problem 10.** By "most sensitive", we mean the following: if you can change one entry of the matrix by a small amount, call it $\varepsilon$, which entry would cause the biggest change in the determinant?

For instance, let's change the 3: the new determinant is $2 \cdot 5 - (3 + \varepsilon)(-1) = 13 - 3\varepsilon$. The original determinant was 13, so we changed it by $-3\varepsilon$. I'll let you figure out which entry causes the most change.

**Problem 11.** Yawn. These are all polynomials in $x$ and $y$, and the procedure is straightforward. I'll do part (e), which obviously should read "$j(x, y, z) = \cdots$". The gradient is $(-4x^3y^4, -4x^4y^3, -4z^3)$. To get a critical point, we must have $z = 0$, and we must have $x$ *or* $y$ equal to zero. Here, both the $x$- and $y$-axis in the $z = 0$ plane are all critical points. Whoa.

The Hessian of this function is

$$\begin{pmatrix} -12x^3y^4 & -16x^3y^3 & 0 \\ -16x^3y^3 & -12x^4y^2 & 0 \\ 0 & 0 & -12z^2 \end{pmatrix}$$

Our critical points all have $z = 0$, but at such points the Hessian has a column of zeros, and hence one of the determinants in Sylvester's Theorem is zero, and we have no conclusion.

More analysis is needed to classify all the critical points. Note that the maximum possible value of this function is 2, since all the 4th powers are always nonnegative. In general, if $z \neq 0$, it's a saddle point. If $z = 0$, you get "weak maxima": if you move in any direction, the value of the function either stays the same or decreases. Think of $f(x, y) = x^2$.

**Problem 12.** Minimize the function $x^2 + y^2 + z^2$ subject to the constraint $3x - 4y - z = 24$. We need to find when the respective gradient vectors $(2x, 2y, 2z)$ and $(3, -4, -1)$ are parallel. You'll find that

$$x = 3k/2, y = 2k, z = -k/2.$$

If we need $(x, y, z)$ to be on that plane, we must have $k = 24/13$. Then you can find the particular point.

**Problem 13.** Same idea: the gradient vectors are $(x/2, 2y)$ and $(2x, 2y)$. For those to be parallel, we must have $x = 0$. There are two such points in our constraint: $y = 1$ and $y = -1$. Both those points maximize $f(x, y)$.