1. Find the derivative of $(g \circ f)$ at the point $(0,1,0)$ if $g(x,y) = (x^3, y)$ and $f(x,y,z) = (4x + y + z^2, xz)$.

   Ans.
   $$D(g \circ f)(0,1,0)(x,y,z) = (12x + 3y, 0)$$

2. Let $g(x,y) = \arctan(y/x)$, and let
   $$F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$$
   so $F = \nabla g$. (You do not need to check that $F = \nabla g$.)

   (a) At $(1,2)$, find the direction in which g is increasing most rapidly. What angle does this direction form with the positive x-direction?

   (b) What is the rate of increase in that direction?

   (c) Suppose the positive y-axis represents north, and the positive x-axis represents east. What is a direction vector $\mathbf{u}$ in the southeast direction? (Remember, a direction vector is a unit vector, i.e. it has a length of 1.) What is the directional derivative of g in the direction of $\mathbf{u}$ at the point (1,2)? If g represents elevation, is this uphill or downhill?
   (d) If g represents elevation, in what direction should one go from the point (2,1) to stay at the same height (i.e. neither uphill or downhill)?

   Ans.

   (a) The direction is $F(1,2) = (-2/5, 1/5)$.

   (b) $1/\sqrt{5}$

3. Let $x^2 + y^2/4 + z^2/9 = 3$, so z is defined implicitly as a function of x and y.

   (a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1,2,3)$.

   (b) Find an equation for the tangent plane to the graph of the equation at the point $(1,2,3)$. Use your answer to (a), and give your answer in the form $z = Ax + By + C$.

   Ans.

   (a) At $(1,2,3)$, $\frac{\partial z}{\partial x} = -3$ and $\frac{\partial z}{\partial y} = -3/2$.

   (b) $z = -3x - 3y/2 + 9$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with $f(5,4) = (8,2)$ and
   $$Jf(5,4) = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$$

   (a) Accurately estimate $f(5.05, 3.96)$.

   (b) Find the directional derivative of f_1 in the direction of the vector $(5,-4)$. (Here we're writing $f = (f_1, f_2)$.)

Ans.

(a) Using the linear approximation, $f(5.05, 3.96) \cong (7.83, -2.16)$.

(b) The directional derivative is approximately $-0.17/\sqrt{0.41} \cong -0.2655$.

5. Let $f(x, y, z) = 0$ define each variable implicitly as a differentiable function of the other two. Prove that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

Ans.

Ask us for help with this one; use our formulas for these partial derivatives. Everything cancels out except for three -1's, leaving $(-1)^3 = -1$.