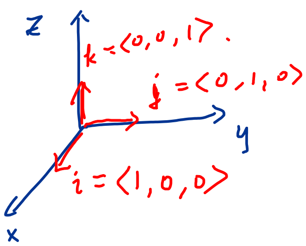
Quick Reivew on sections 1.1, 1.2

$\vec{a} = \langle 1, 2, 3 \rangle$

$\|\vec{a}\| = \sqrt{1^2+2^2+3^2}$

unit vector $= \dfrac{\vec{a}}{\|\vec{a}\|} = \dfrac{\langle 1,2,3\rangle}{\sqrt{14}}$.

- A vector: a quantity with direction and magnitude.
- Notation of a vector: use "bold" **a** or $\vec{a}$.
- Given a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$. The length of a vector $\vec{a}$ is denoted by $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- Standard Basic vectors:



$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

EX: $\vec{a} = \langle 3, -1, 2 \rangle$

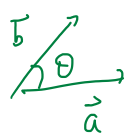
$= 3i - j + 2k$.

Then $\vec{a} = \langle a_1, a_2, a_3 \rangle$ can be written as $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

- (Inner product or Dot product) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$. The Dot product formula is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

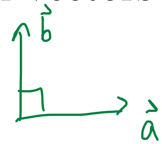
- $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$, where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.



$\theta = 90^\circ$,

$\cos 90^\circ = 0$.

$\vec{a} \cdot \vec{b} = 0$.

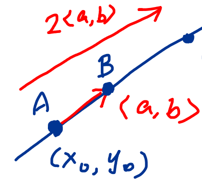


Review: Parametrization of a line

In two dimensions, a **parametrization** for a line through point (x_0, y_0), in direction of $\langle a, b\rangle$ is

$B = (x_0, y_0) + 1\langle a, b\rangle$

$$(x, y) = (x_0, y_0) + t(a, b) \text{ for } -\infty < t < \infty.$$



Its **parametric equation** is

$$x = x_0 + at$$
$$y = y_0 + bt$$

$C = (x_0, y_0) + 2\langle a, b\rangle$.

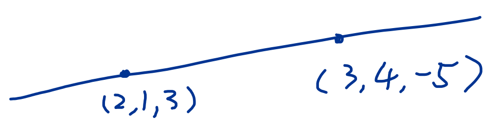
In three dimensions, a **parametrization** for a line through point (x_0, y_0, z_0), in direction of $\langle a, b, c\rangle$ is

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \text{ for } -\infty < t < \infty.$$

Its **parametric equation** is

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Example:



1. Find a parametrization of the line passes through points $(2, 1, -3)$ and $(3, 4, -5)$.

   point : $\langle 2, 1, -3\rangle$

   direction: $\langle 3, 4, -5\rangle - \langle 2, 1, -3\rangle = \langle 1, 3, -2\rangle$.

   A parametrization: $(x, y, z) = (2, 1, -3) + t(1, 3, -2)$,

   $-\infty < t < \infty$.

2. Find a parametrization for the line segment between the points $(2, 1, -3)$ and $(3, 4, -5)$.

   $(x, y, z) = (2, 1, -3) + t(1, 3, -2)$

   $0 \le t \le 1$.

1.3 Matrices, Determinants, Cross Product

A **matrix** is simply a rectangle array of numbers, such as,

$$\begin{array}{cc} & \text{columns} \\ \text{rows} & \begin{bmatrix} 1 & 0.7 & 10 \\ \pi & 6 & 0 \end{bmatrix} \end{array}$$

The above matrix is a $\underset{\text{row}}{\underline{2}} \times \underset{\text{column.}}{\underline{3}}$ matrix because it has 2 rows and 3 columns

- 2×2 matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow \text{1st row, 2nd column}$$

The determinant: We denote it as $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$ or $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3×3 matrices:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant: We denote it as $\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)$ or $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array}$$

$$= \boxed{a_{11}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Example: Evaluate $\begin{vmatrix} 1 & 2 \\ 4 & 11 \end{vmatrix}$ $= 1(11) - 2 \times 4 = 11 - 8 = 3.$

Also evaluate $\det\left(\begin{bmatrix} 2 & 1 \\ 11 & 4 \end{bmatrix}\right)$ $= -3.$

Can you guess what is the general rule for interchanging two columns or two rows?

Example: Evaluate $\det\left(\begin{bmatrix} 2 & -1 & 0 \\ 4 & 3 & 2 \\ 3 & 0 & 1 \end{bmatrix}\right)$

$$= 2\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & 3 \\ 3 & 0 \end{vmatrix}.$$

$$= 2(3-0) - (-1)(4-6) + 0.$$

$$= 4. \quad \#$$

§**Cross Product** We define the **Cross product** or vector product for two 3-dimensional vectors, $\vec{a} = \langle a_1, a_2, a_3\rangle$, $\vec{b} = \langle b_1, b_2, b_3\rangle$ by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k.$$

$i = \langle 1, 0, 0\rangle$

$j = \langle 0, 1, 0\rangle$

$k = \langle 0, 0, 1\rangle$

$$= (a_2 b_3 - a_3 b_2)\, i - (a_1 b_3 - a_3 b_1)\, j + (a_1 b_2 - b_1 a_2)\, k$$

$$= \langle a_2 b_3 - a_3 b_2,\ a_1 b_3 - a_3 b_1,\ a_1 b_2 - b_1 a_2\rangle.$$

a vector

NOTE: $\vec{a}\cdot\vec{b}$ is scalar number.

Example: Let $\vec{a} = \langle 1, 2, 0\rangle$, $\vec{b} = \langle -1, 3, 2\rangle$. Find $\vec{a}\times\vec{b}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} k.$$

$$= (4, -2, 5).\ \#$$

§Cross Product

Properties: Let $\vec{a}, \vec{b}, \vec{c}$ are vectors and α, β are constants.

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

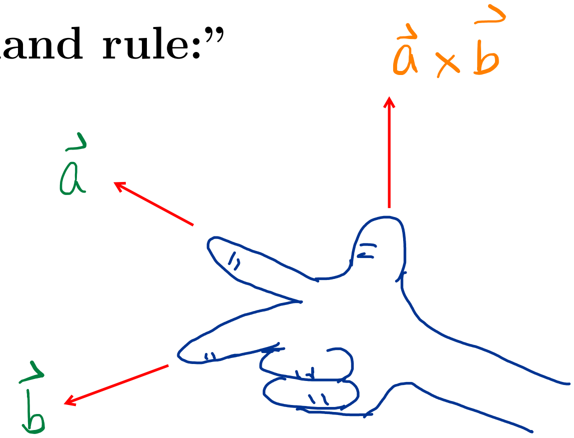
2. $\vec{a} \times (\alpha\vec{b} + \beta\vec{c}) = \alpha(\vec{a} \times \vec{b}) + \beta(\vec{a} \times \vec{c})$.

   α, β : constant.

3. **Geometric meaning:**

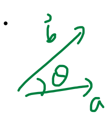
   $\vec{a} \times \vec{b}$ is a vector **perpendicular** to $\vec{a}$ and $\vec{b}$.

   "Right hand rule:"

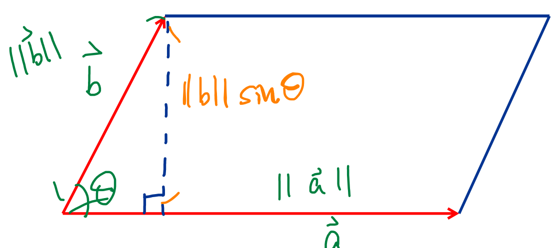


   NOTE:
   $(\vec{a} \times \vec{b}) \perp \vec{a}$
   $(\vec{a} \times \vec{b}) \perp \vec{b}$

4. $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin(\theta)$.



5. Recall that the area of the parallelogram = Base $\times$ Height.
   Consider the parallelogram spanned by vectors $\vec{a}$ and $\vec{b}$.



   area of parallelogram
   $= \text{Base} \times \text{Height}$
   $\quad \|\vec{a}\| \qquad \|b\| \sin\theta$,

   Thus, $\|\vec{a} \times \vec{b}\| =$ area of parallelogram $= \|\vec{a}\| \|\vec{b}\| \sin\theta$.
   $= \|\vec{a} \times \vec{b}\|$