**Math 2374 Spring 2018 - Week 2**

# Quick Reivew from Last week

- Standard Basic vectors: in 3D,

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

(Dot)

- (Inner product) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

  Also, $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$, where $\theta$ is the angle between vectors $\vec{a}$ and $\vec{b}$.

- (Cross product) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| |\sin(\theta)|.$$

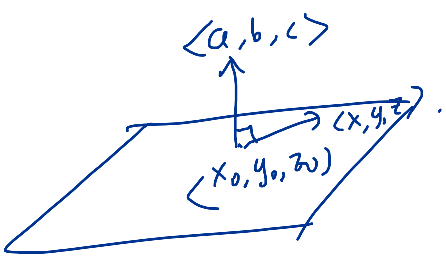
- (Parametric equations of a line) In 3D, a line through point $(x_0, y_0, z_0)$, in direction of $\langle a, b, c \rangle$

(parametric equation)

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

- (Equations of planes) If point $(x_0, y_0, z_0)$ in the plane and $\langle a, b, c \rangle$ is a normal vector for the plane, then the equation of the plane is

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0.$$

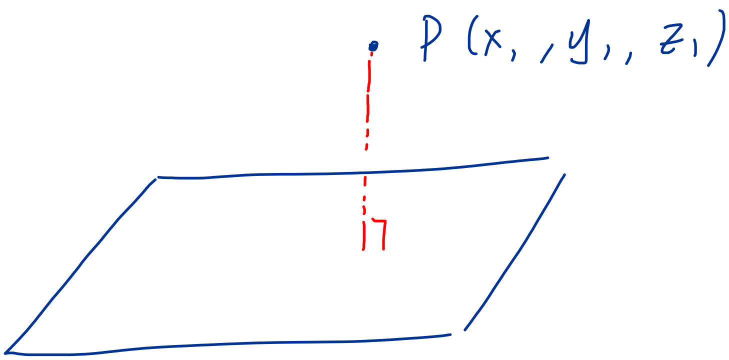



Quiz 1: 1.1, 1.2, 1.3, 1.5.

## §Distance from a point to a plane

The distance from a point $P = (x_1, y_1, z_1)$ to the plane $ax + by + cz + D = 0$ is

$$\text{dist} = \frac{|ax_1 + by_1 + cz_1 + D|}{\sqrt{a^2 + b^2 + c^2}}$$




$ax + by + cz + D = 0.$

NOTE: $\langle a, b, c \rangle$ is normal vector.

**Example 9:**
Find the distance from the point $(1, 1, -2)$ to the plane $12x + y + 2z = -4$.

$$\text{dist} = \frac{|12(1) + 1 + 2(-2) + 4|}{\sqrt{12^2 + 1 + 2^2}}$$

$$= \frac{|12 + 1 - 4 + 4|}{\sqrt{144 + 1 + 4}}$$

$$= \frac{13}{\sqrt{149}}.$$

#

## 1.5 $n$-dimensional Euclidean Space

In section 1.1 and 1.2, we have studied the space $\mathbb{R}^1, \mathbb{R}^2$, and $\mathbb{R}^3$. For example, we think of $\mathbb{R}^3$ as a set of triples (vectors)

$$\langle x, y, z \rangle$$

where $x, y$ and $z$ are real numbers. We call $\mathbb{R}^3$ is 3-dimensional Euclidean space.

For generalization, we denote $n$-dimensional Euclidean space by $\mathbb{R}^n$ whose elements are vectors written as

$$x = \langle x_1, x_2, \cdots, x_n \rangle$$

where $x_i$ is a real number.

Example,

$$\mathbf{a} = (1,\ 6,\ -23,\ 0.11,\ \pi)$$

is a vector in $\mathbb{R}^5$.

Now we want to study some properties that are analogous to those introduced in previous sections for $\mathbb{R}^1, \mathbb{R}^2$, and $\mathbb{R}^3$.

In $\mathbb{R}^n$, we have

- (Addition)

$$\langle x_1, x_2, \cdots, x_n \rangle + \langle y_1, y_2, \cdots, y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n \rangle$$

- (Scalar multiplication) For any real number $\alpha$,

$$\alpha \langle x_1, x_2, \cdots, x_n \rangle = \langle \alpha x_1, \alpha x_2, \cdots, \alpha x_n \rangle$$

## §Standard basis vectors of $\mathbb{R}^n$:

$$
\begin{aligned}
\mathbf{e_1} &= \langle 1, 0, \cdots, 0 \rangle \\
\mathbf{e_2} &= \langle 0, 1, \cdots, 0 \rangle \\
&\vdots \\
\mathbf{e_n} &= \langle 0, 0, \cdots, 1 \rangle.
\end{aligned}
$$

Then we can write a vector $x$

$$x = \langle x_1, x_2, \cdots, x_n \rangle = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \cdots + x_n\mathbf{e_n}$$

Ex: $\vec{a} = \langle 1, 10, -10, \pi \rangle$ in $\mathbb{R}^4$.

$= 1e_1 + 10e_2 - 10e_3 + \pi e_4$.

## §Inner(Dot) product: Two vectors in $\mathbb{R}^n$

$$\vec{u} = \langle u_1, \cdots, u_n \rangle, \quad \vec{v} = \langle v_1, \cdots, v_n \rangle.$$

The **inner product** of $u$ and $v$ is

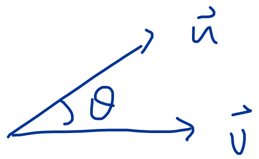
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

$\vec{a} = \langle 1, 10, -10, \pi \rangle$. $\|\vec{a}\| = \sqrt{1^2 + 10^2 + (-10)^2 + \pi^2}$.

**Fact.** *1. Length of vector $\vec{u}$ is*

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_1^2 + \cdots + u_n^2}$$

*2. $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos(\theta)$ where $\theta$ is the angle of vectors $\vec{u}$ and $\vec{v}$.*



## §Introduction to general matrices

An $m \times n$ matrix is any array of $mn$ numbers with $m$ rows and $n$ columns:

row column.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Notation: $A = [a_{ij}]$.

**Example 1.**

$$B = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$$

*is* $2 \times 3$ *matrix.*

row column.

**§Matrix Algebra.** Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. Let $\alpha$ be a constant.

- Sum : $A + B = [a_{ij} + b_{ij}]$
- Scalar multiple: $\alpha A = [\alpha a_{ij}]$

**Example 2.**

2×3 matrix     2×3 matrix     2×2 matrix.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

1. $A + B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 8 \end{bmatrix}$

2. $4A = 4\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 0 \\ 12 & 16 & 4 \end{bmatrix}$.

3. $A - C =$ not defined !

## §Vectors as matrics.

We write an $n-$dimensional vector $\mathbf{x}$ as a $n \times 1$ column matrix:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

We call $\mathbf{x}$ a $n \times 1$ **column vector.**

Example,

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 9\pi \end{bmatrix}.$$

a $4 \times 1$ column vector.

Ex: vector $\vec{x} = \langle 1, 2, 10 \rangle$

$$= \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}, \quad 3 \times 1 \text{ column vector.}$$

- Multiplication: **multiplication between a matrix A and a vector x.** We define this multiplication only for $m \times n$ matrix $A$ and $n \times 1$ column vector $\mathbf{x}$.

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} \langle a_{11}, \dots, a_{1n} \rangle \cdot \vec{x} \\ \langle a_{21}, \dots, a_{2n} \rangle \cdot \vec{x} \\ \vdots \\ \langle a_{m1}, \dots, a_{mn} \rangle \cdot \vec{x} \end{bmatrix}$$

$m \times 1$.

$$\vec{x} = \langle x_1, \dots, x_n \rangle = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

**Example 3.**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad 2 \times 3 \text{ matrix}$$

*and* $\boldsymbol{x} = (1, 0, 2)$. *Compute* $A\boldsymbol{x}$.

$$A x = \overset{2\times 3}{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}} \overset{3 \times 1}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}, \quad 2 \times 1 \text{ matrix.}$$

$$= \begin{bmatrix} \boxed{2} \\ \boxed{5} \end{bmatrix}$$

$$(2,1,0) \cdot (1,0,2) = 2 + 0 + 0 = 2$$

$$(1,1,2) \cdot (1,0,2) = 1 + 0 + 4 = 5.$$

**Multiplication between a matrix A and a matrix B.**

$A = [a_{ik}]$ $m \times n$ matrix; $B = [b_{kj}]$ $n \times p$ matrix.

We say $C = AB$, the product of $A$ and $B$.

Then $C = [c_{ij}]$ is an $\underline{m \times p}$ matrix

$$\begin{bmatrix} \cdots & \boxed{c_{ij}} & \cdots \end{bmatrix} = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

i-row

j-column.

i row

j-column

$c_{ij} = \langle a_{i1}, \ldots, a_{in} \rangle \cdot \langle b_{1j}, \ldots, b_{nj} \rangle.$

**Example 4.**

$2 \times 3$ $\quad$ $3 \times 3$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$2 \times 3$ matrix

$$AB = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$(2,1,0) \cdot (1,0,1)$

$(2,1,0) \cdot (0,2,1) = 2$

$$= \begin{bmatrix} \boxed{2} & \boxed{2} & 5 \\ 3 & 4 & 5 \end{bmatrix}$$

$2 \times 3$.

$3 \times 3$ , $2 \times 3$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

= not defined !

**Example 5.**

$$\overset{2\times 1 \checkmark}{A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \quad \overset{1\times 2 \checkmark}{B = \begin{bmatrix} 2 & 2 \end{bmatrix}}$$

$$AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}_{2\times 2}.$$

$$BA = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}. \quad \#$$

**Properties of Matrices:**

- In general $AB \neq BA$
- If $AB$ and $BC$ are defined, then $(AB)C$, $A(BC)$ are defined. In fact, $(AB)C = A(BC)$.

## §Dot product in matrix notation.

We take the **transpose** of the $n \times 1$ column vector, turning it into a $1 \times n$ row matrix.

For example,

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 9\pi \end{bmatrix}.$$

a $4 \times 1$ column vector. The transpose of $\mathbf{x}$ denoted by

$$\mathbf{x}^T = \begin{bmatrix} 3 & 4 & 0 & 9\pi \end{bmatrix}.$$

Then the inner product of two vectors

$$\mathbf{u} = (u_1, \cdots, u_n), \quad \mathbf{v} = (v_1, \cdots, v_n)$$

is equivalent to a matrix multiplication: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$

$n \times 1$ column vector

$$\mathbf{u}^T\mathbf{v} = \underbrace{\overbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}}^{\text{transpose of } u} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{\text{multiplication of 2 matrices.}} = \begin{bmatrix} u_1 v_1 + \cdots + u_n v_n \end{bmatrix} \quad 1 \times 1 \text{ matrix,}$$