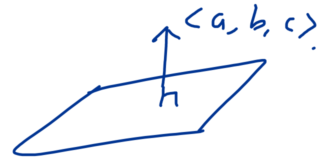
Math 2374 Spring 2018 - Week 3

Quick Reivew from last week



- The distance from a point $P = (x_1, y_1, z_1)$ to the plane $ax + by + cz + D = 0$ is

normal vector $\langle a, b, c \rangle$.

$$\frac{|ax_1 + by_1 + cz_1 + D|}{\sqrt{a^2 + b^2 + c^2}}$$

- Multiplication of matrices.

- Linear transformations :

  – A 2-dimensional linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

  $$T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  where a, b, c, d are numbers.
  If $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$, then T maps **parallelograms** onto **parallelograms** and vertices into vertices.

  – If A is a 3×3 matrix with $\det(A) \neq 0$, then a 3-dimensional linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T\mathbf{x} = A\mathbf{x}$ maps **parallelepipeds** onto **parallelepipeds**.

2.1 The Geometry of Real-Valued Functions

In this section, we will develop methods for visualizing a function.

§Functions:

Let f be a function which assigns to each vector $x = \langle x_1, \cdots, x_n \rangle$ in a subset U of $\mathbb{R}^n$, a unique vector $f(x)$ in $\mathbb{R}^m$.

We call U is the **domain** of f. We denote this function f by

$$f : U \subset \mathbb{R}^n \to \mathbb{R}^m$$

to indicate that f maps from U into $\mathbb{R}^m$. Here $\subset$ means "is subset of".

We denote the component functions of $f(x)$ as follows:

$$f(x) = (f_1(x), \cdots, f_m(x)).$$

- If $m = 1$, then we call f is a **scalar**-valued function.
- If $m > 1$, then we call f is a **vector**-valued function.

Example 1. *1.* $f(\underbrace{x, y}_{\text{2 variables}}) = \overbrace{x + y}^{\text{scalar}}$ *is a function of two variables and f gives a value.*

For example, f maps $(2, 1)$ to a number 3, that is, $f(2, 1) = 3$.

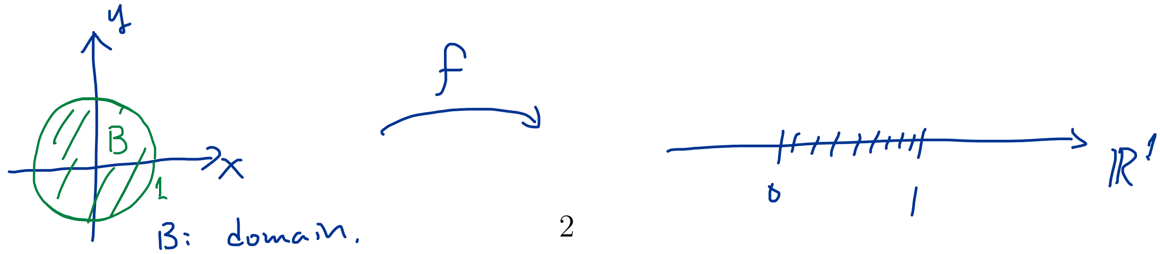
Thus, f maps $\mathbb{R}^2$ to $\mathbb{R}$. In addition, f is a scalar-valued function.

$f : \mathbb{R}^2 \to \mathbb{R}^1$

2. $f(\overbrace{x, y, z}^{\mathbb{R}^3}) = \overbrace{(x^2 y,\ \cos(z) + e^x)}^{\mathbb{R}^2}$. *So $f : \mathbb{R}^3 \to \mathbb{R}^2$ and f is a vector-valued function.*

EX: $f(x, y) = x^2 + y^2$ on the unit disk, B, ($x^2 + y^2 \leq 1$)

$f : B \subset \mathbb{R}^2 \to \mathbb{R}^1$



One way to visualize functions is through their graphs.(Prelecture study in math insight)

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$. Then its graph is the surface formed by the set of points (x, y, z) where $z = f(x, y)$.

scalar-valued function.

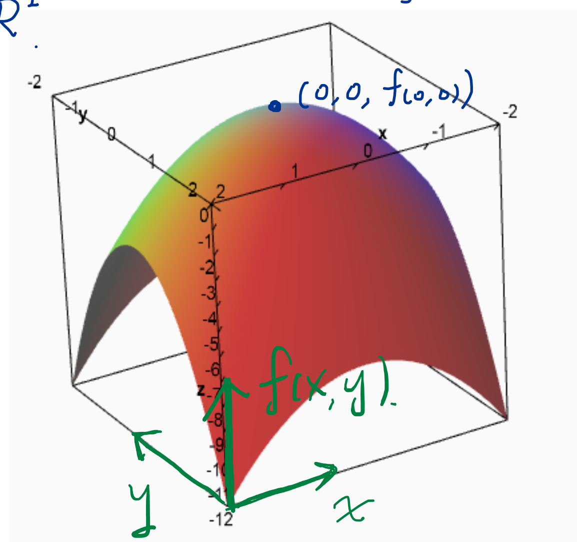
Example 2. *$f(x, y) = -x^2 - y^2$ with the domain defined by $-2 \le x \le 2$ and $-2 \le y \le 2$. The graph of all points $(x, y, f(x, y))$ is an elliptic paraboloid.*

$(x, y, -x^2-y^2)$

$f : [-2,2] \times [-2,2] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^1$.

f has maximum at $(0,0)$

$f(0,0) = 0$.



If we consider $f : \mathbb{R}^3 \to \mathbb{R}$. Then its graph is the surface formed by the set of points (x, y, z, t) where $t = f(x, y, z)$. This surface is in 4 dimensions, therefore it would be difficult to imagine such a graph.

§Level Sets:

Another way to visualize functions is through **level set**, that is a subset of the domain of function f on which f is a constant. That is,

such that

$$\text{Level set is } \{x | f(x) = c\}$$

the set of

(that is, the set of the point x such that $f(x) = c$)

where c is constant

(scalar-valued function)

- The level sets for $f : \mathbb{R}^2 \to \mathbb{R}^1$ are curves.
  We call **level curves** or **level contours**.

Example 3. $f(x, y) = x^2 + y^2$. *Describe the level sets of f.*

levels sets:

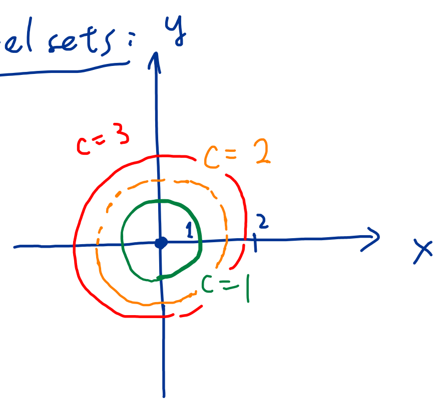
$c = 0$, $f(x, y) = 0$, then $(x, y) = (0, 0)$.

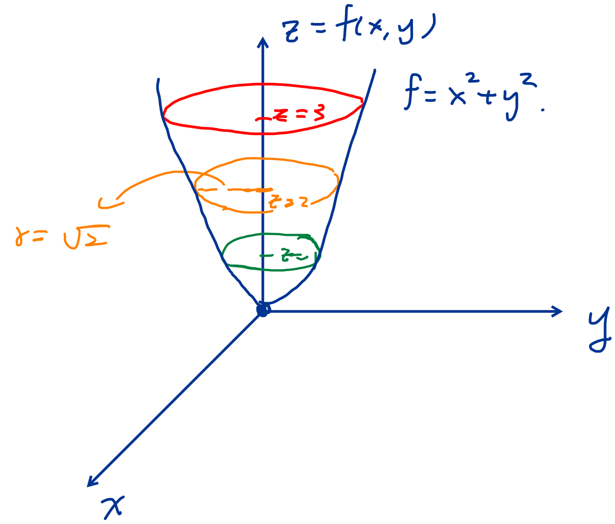
$c = 1$, $f(x, y) = 1$, $x^2 + y^2 = 1$, unit circle.

$c = 2$, $f(x, y) = 2$, $x^2 + y^2 = 2$, circle with radius $\sqrt{2}$.

$\vdots$

Level sets:





Graph of f.

Example 4. *Let* $f(x,y) = x^2 - y^2$. *Study the level curves of* f.

level sets

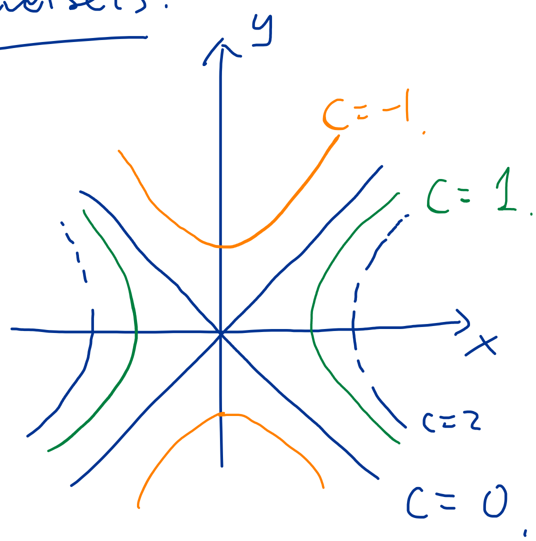
$c=0$, $f(x,y) = 0$, $x^2 - y^2 = 0$, $x = \pm y$.

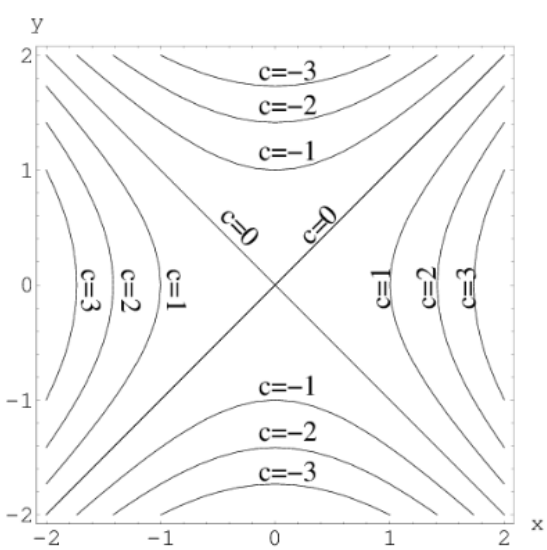
$c=1$, $f(x,y) = 1$, $x^2 - y^2 = 1$, hyperbola.

$c=2$, $f = 2$, $x^2 - y^2 = 2$

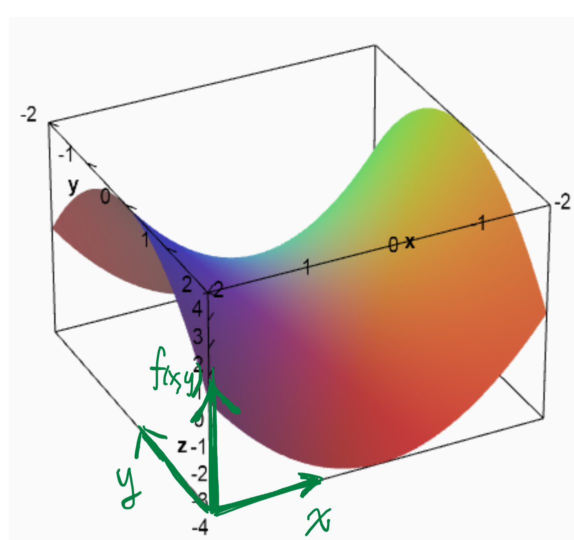
$c=-1$, $f = -1$, $x^2 - y^2 = -1$,

level sets:





Graph of f.



- The level sets for $f : \mathbb{R}^3 \to \mathbb{R}^1$ are surfaces.
  We call **level surfaces**.

Example 5. $f(x, y, z) = x^2 + y^2 + z^2$. *Describe the level sets of f.*

level sets

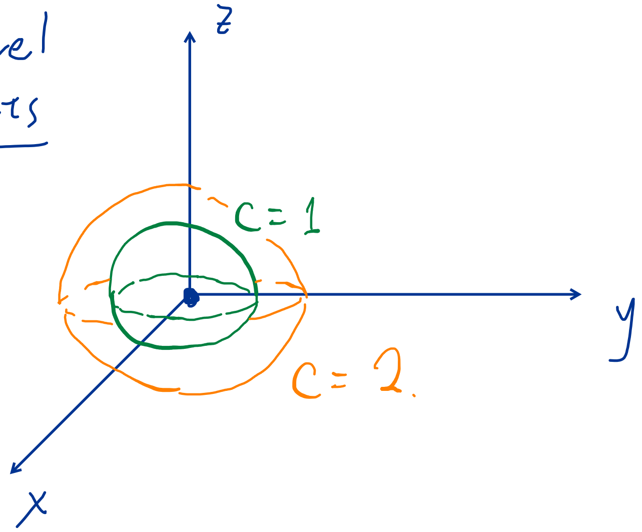
$c = 0$, $f(x, y, z) = 0$, $(x, y, z) = (0, 0, 0)$.

$c = 1$, $f(x, y, z) = 1$, $x^2 + y^2 + z^2 = 1$, unit sphere

$c = 2$, $f(x, y, z) = 2$, $x^2 + y^2 + z^2 = 2$, sphere with radius $\sqrt{2}$.

$\vdots$

level sets



NOTE: Graph of f is in 4 dimensions.

2.3 Differentiation

In section 2.1, we have discussed some methods for visualizing a function, e.x., drawing the level sets, sections.

From Calculus 1, we knew that the derivative of a function can tell us many things about this function such as locating maxima/minima, and rates of change.

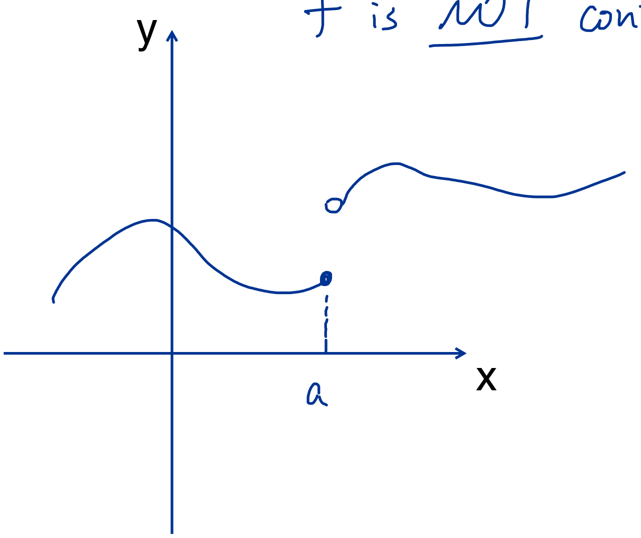
In Calculus 1, we have learned, for function $f : \mathbb{R}^1 \to \mathbb{R}^1$,

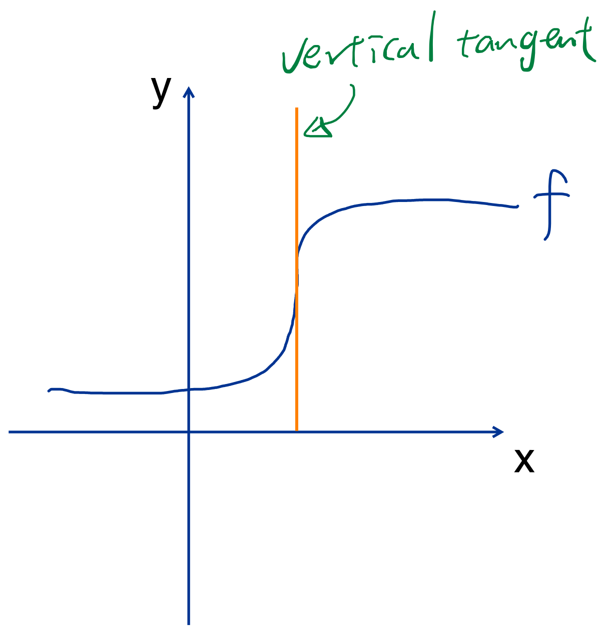
- Continuous: No break in the graph of f.

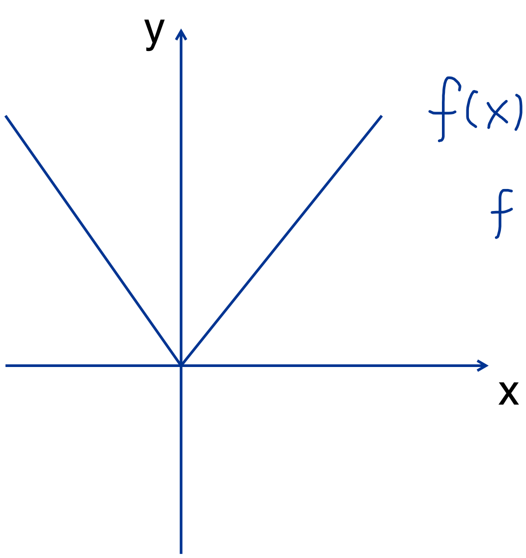
- Differentiability: f is continuous, no corner, no vertical tangent line.

Example:

f is NOT continuous at $x = a$







$f(x) = |x|$

f is NOT differentiable at $x = 0$.

§What does it mean to take "derivative" of $f : \mathbb{R}^n \to \mathbb{R}^1$ for $n > 1$?

the intersection of the graph of f and $y =$ constant

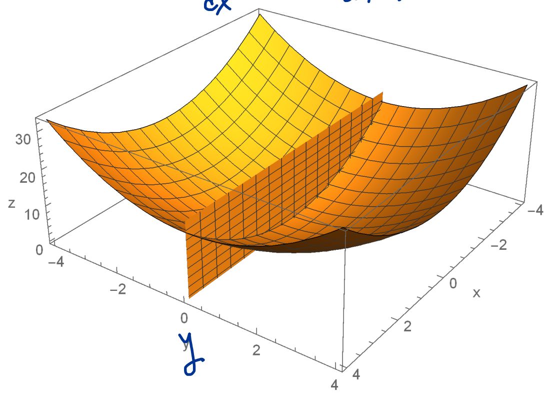
- When y is constant, you walk on this <u>section</u>, then find the slope as x changes.

Example 1. *Let* $z = f(x, y) = 1 + x^2 + y^2$.

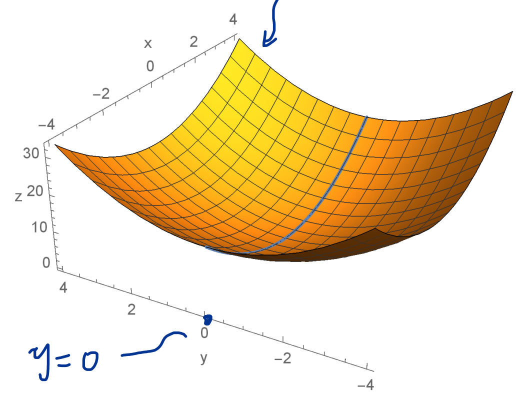
When $y = 0$, then $z = 1 + x^2$ is a parabola on the plane $y = 0$.

$z = f(x, 0) = 1 + x^2$.

$\frac{dz}{dx} = 2x$.



graph of $f(x,y)$



$y = 0$

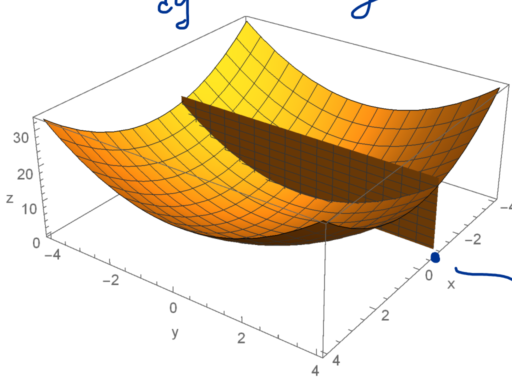
The blue curve above is the intersection of graph of f and $y = 0$.

Similarly,

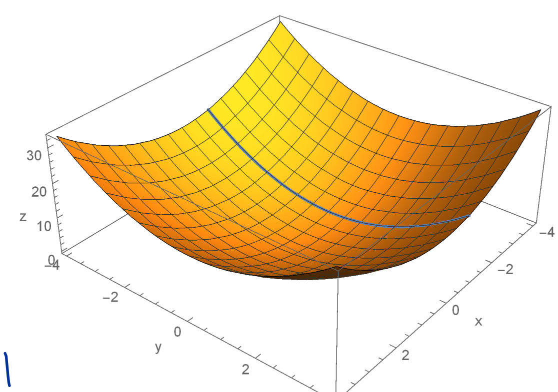
- When x is constant, you walk on this <u>section</u>, then find the slope as y changes. Ex: $x = -1$.

$z = f(-1, y) = 2 + y^2$.

$\frac{dz}{dy} = 2y$.



$x = -1$



The blue curve above is the intersection of graph of f and $x = -1$.

§Partial derivatives of f

Let's first consider $f : \mathbb{R}^2 \to \mathbb{R}^1$.

1. **Partial derivatives** of f with respect to x: (f_x or $\frac{\partial f}{\partial x}$)

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

We also denote $\frac{\partial f}{\partial x}(x, y)$ by $f_x(x, y)$.

2. **Partial derivatives** of f with respect to y: (f_y, or $\frac{\partial f}{\partial y}$).

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

We also denote $\frac{\partial f}{\partial y}(x, y)$ by $f_y(x, y)$.

More general definition: $f : \mathbb{R}^n \to \mathbb{R}^1$.

Then j**-th parital derivative** of f, for $j = 1, \cdots, n$, is a function

$$\frac{\partial f}{\partial x_j} : \ \mathbb{R}^n \to \mathbb{R}^1$$

defined by

$$\frac{\partial f}{\partial x_j}(x_1, \cdots, x_n) = \lim_{h \to 0} \frac{f(x_1, \cdots, x_j + h, \cdots, x_n) - f(x_1, \cdots, x_j, \cdots, x_n)}{h}$$

if the limit exist.

Recall from Calculus 1, for $f : \mathbb{R}^1 \to \mathbb{R}^1$.

- Product rule:
$$\frac{d}{dx}(fg) = f'g + fg'$$
- Quotient rule:
$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$$
- Chain rule:
$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Example 2. *Let* $f(x, y, z) = (1 + z^2)e^{\cos(xy^2)} + \underline{7\cos(z)y^3}$. *Find partial derivatives* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, *and* $\frac{\partial f}{\partial z}$.

(view y, z as constant)

$\frac{\partial}{\partial x}(7\cos(z)y^3)$

$$\frac{\partial f}{\partial x} = (1+z^2)e^{\cos(xy^2)}\frac{\partial}{\partial x}(\cos(xy^2)) + 0.$$

$$= (1+z^2)e^{\cos(xy^2)}\underline{(-\sin(xy^2)\,y^2)}. \quad \#$$

To be continued!