Recall from Calculus 1, for $f : \mathbb{R}^1 \to \mathbb{R}^1$.

- Product rule:
$$\frac{d}{dx}(fg) = f'g + fg'$$
- Quotient rule:
$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$$
- Chain rule:
$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Example 2. *Let $f(x, y, z) = (1 + z^2)e^{\cos(xy^2)} + 7\cos(z)y^3$. Find partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.*

(view y, z as constant)

$$\frac{\partial f}{\partial x} = (1+z^2)e^{\cos(xy^2)}\frac{\partial}{\partial x}(\cos(xy^2)) + 0 .$$

$\frac{\partial}{\partial x}(7\cos(z)y^3)$

$$= (1+z^2)\, e^{\cos(xy^2)} \underline{(-\sin(xy^2)\; y^2)} . \quad \#$$

view x, z as constants

$$\frac{\partial f}{\partial y} = (1+z^2)\, e^{\cos(xy^2)}(-\sin(xy^2)\, 2xy) + 21\cos(z)\, y^2 .$$

$$\frac{\partial f}{\partial z} = 2z\, e^{\cos(xy^2)} + (-7\sin z)\, y^3 .$$

NOTE:

$$Df = [\, f_x \quad f_y \quad f_z \,] .$$

$$\boxed{\begin{aligned} f_x &= \frac{\partial f}{\partial x} \\ f_y &= \frac{\partial f}{\partial y} \\ f_z &= \frac{\partial f}{\partial z} \end{aligned}}$$

§Matrix of partial derivatives

→ f is scalar-valued function.

For the derivative of $f : \mathbb{R}^n \to \mathbb{R}^1$, its **matrix of partial derivatives** is

$$\mathbf{D}f(a) = \left[\frac{\partial f}{\partial x_1}(a) \ \cdots \ \frac{\partial f}{\partial x_n}(a)\right], \quad 1 \times n \text{ matrix.}$$

vector-valued function.

If $f : \mathbb{R}^n \to \mathbb{R}^m$, then its **matrix of partial derivatives** is

$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$

$$\mathbf{D}f(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}, \quad m \times n \text{ matrix}$$

Example 3. *(EX6 in page 111) Compute matrix of partial derivatives for function* $f(x, y, z) = (\underbrace{ze^x}_{f_1}, \underbrace{-ye^z}_{f_2})$. Df is 2×3 matrix.

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}_{2\times 3}$$

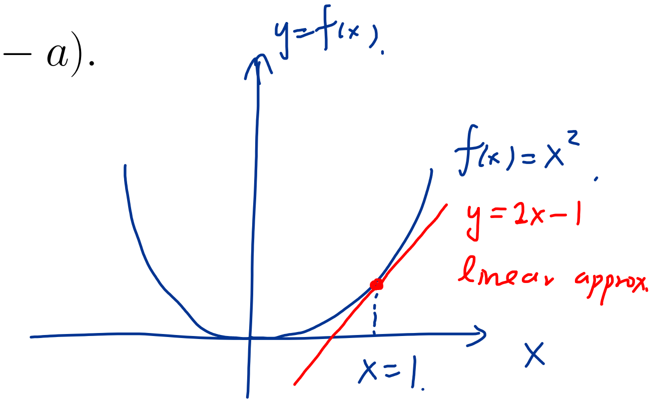
$$= \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$$

$\neq$

To motivate the definition of differentiability, let us start from observing the linear approximation if f is "smooth enough".

Recall in Calculus 1, we say the tangent line at $(a, f(a))$ is an linear approximation to the curve $f(x)$ near $x = a$, that is,

$$L(x) = f(a) + f'(a)(x - a).$$



Ex: $f(x) = x^2$.

$f'(x) = 2x$

tangent line (linear approximation)

at $x = 1$ is

$y = f(1) + f'(1)(x-1)$.

$= 1 + 2(x-1) \Rightarrow y = 2x - 1$.

§Linear Approximation

Now for function $f : \mathbb{R}^2 \to \mathbb{R}^1$, we call the plane tangent to the graph of f at point (x_0, y_0) is the **linear approximation** of f near (x_0, y_0) which can be expressed as follows:

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0).$$

§Tangent plane

We now formally introduce the plane tangent to the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}^1$.

If f is differentiable at (x_0, y_0), then the **tangent plane** of the graph of f at $(x_0, y_0, f(x_0, y_0))$ is

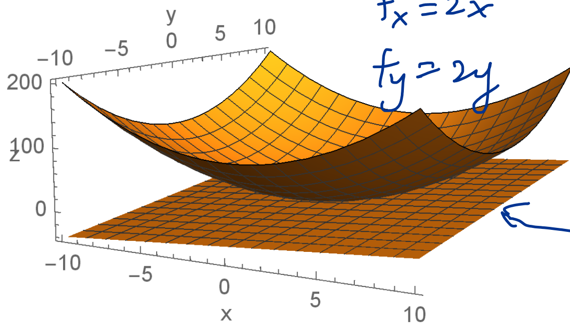
$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0) \tag{1}$$

$$= f(x_0, y_0) + \mathbf{D}f(x_0, y_0)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \tag{2}$$

Normal vector : $\langle f_x(x_0,y_0), f_y(x_0,y_0), -1 \rangle$.

Recall: $Df(x,y) = \begin{bmatrix} f_x(x,y) & f_y(x,y) \end{bmatrix}_{1\times 2}$.

EX: $f(x,y) = 1 + x^2 + y^2$. Linear approximation of f near $(1,1)$.



$f_x = 2x$

$f_y = 2y$

Linear approximation.

$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$= 3 + 2(x-1) + 2(y-1)$

Example 4. *Let* $f(x,y) = (1+y)e^{2x+3y}$.

1. *Find* $Df(1,0)$.

$$Df = [\,f_x \quad f_y\,]$$

$$= [\,2(1+y)e^{2x+3y} \quad e^{2x+3y} + 3(1+y)e^{2x+3y}\,].$$

$$Df(1,0) = [\,2e^2 \quad e^2 + 3e^2\,]$$

$$= [\,\underbrace{2e^2}_{f_x(1,0)} \quad \underbrace{4e^2}_{f_y(1,0)}\,]. \#$$

2. *Find the equation for the tangent plane at* $(x,y) = (1,0)$.

Tangent plane is

$$z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)\,y$$

$$= e^2 + 2e^2(x-1) + 4e^2 y.$$

$$z = 2e^2 x + 4e^2 y - e^2$$

(NOTE: normal vector is $\langle 2e^2, 4e^2, -1 \rangle$)

3. Find a linear approximation of the function f near the point $(1, 0)$. Also use it to approximate the value of $f(0.9, 0.01)$. [1]

A linear approximation

$$L(x, y) = e^2 + 2e^2(x-1) + 4e^2 y.$$

$$f(0.9, 0.01) \sim L(0.9, 0.01)$$

$$= e^2 + 2e^2(0.9-1) + 4e^2(0.01)$$

$$= e^2(1 - 0.2 + 0.04)$$

$$= 0.84\, e^2 \quad (\approx 6.206807)$$

[1] $f(0.9, 0.01) \approx 6.29623$

§Differentiability for functions $f : \mathbb{R}^2 \to \mathbb{R}^1$

Roughly speaking, the definition of differentiability means the linear approximation

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

is a "good" approximation of f near (x_0, y_0).

Formal definition is as follows:

We say f is **differentiable** at (x_0, y_0) if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (x_0, y_0) and if

$$\frac{f(x, y) - \overbrace{\left(f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)\right)}^{\text{(linear approximation).}}}{\|(x, y) - (x_0, y_0)\|} \to 0$$

as $(x, y) \to (x_0, y_0)$.

§Differentiability for functions $f : \mathbb{R}^n \to \mathbb{R}^m$

General definition is as follows:

We say f is **differentiable** at x_0 in $\mathbb{R}^n$ if all partial derivatives of f exists at x_0 and

$$\frac{f(x) - \overbrace{\left(f(x_0) + \mathbf{D}f(x_0)(x - x_0)\right)}^{\text{linear approximation}}}{\|x - x_0\|} \to 0$$

as $x \to x_0$.

Here

(matrix of partial derivatives)

$$\mathbf{D}f(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \quad m \times n \text{ matrix}$$

§Gradient of f

For the derivative of $f : \mathbb{R}^n \to \mathbb{R}^1$, its matrix of partial derivatives is

$$\mathbf{D}f(x) = \left[\frac{\partial f}{\partial x_1} \ \cdots \ \frac{\partial f}{\partial x_n}\right], \quad 1 \times n \text{ matrix.}$$

If we write it as a vector form

$$\left\langle \frac{\partial f}{\partial x_1}, \ \cdots, \ \frac{\partial f}{\partial x_n} \right\rangle,$$

then we call it the **gradient of** f, denoted by ∇f or $\text{grad} f$. (We will revisit it in section 2.6.)

Example 5. *Let*

$$f(x, y, z) = xe^{\cos(y)} + z^2.$$

Find ∇f*.*

$$\nabla f = (f_x , f_y , f_z)$$

$$= (e^{\cos y} , -x \sin y \, e^{\cos y} , 2z \rangle . \#$$

NOTE : we will revisit it later!

§Some Facts

Fact. *If f is differentiable at x_0, then f is continuous at x_0.*

Fact. $f : \mathbb{R}^m \to \mathbb{R}^n$*. Suppose the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous in a neighborhood of x_0. Then f is differentiable at x_0.*