Math 2374 Spring 2018 - Week 4

Quick Reivew from last week

Sec. 2.1: Level sets of functions, and graph of functions.

Sec. 2.3:

- Partial derivatives: $\frac{\partial f}{\partial x}(x,y) = f_x(x,y)$, ...
- If $f : \mathbb{R}^n \to \mathbb{R}^m$, then its **matrix of partial derivatives** is

$f = (f_1(x), \ldots, f_m(x))$

$$\mathbf{D}f(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}, \quad m \times n \text{ matrix}$$

- If f is differentiable at (x_0, y_0), then the **tangent plane** of the graph of f at $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y - y_0) \tag{1}$$

$$= f(x_0,y_0) + \mathbf{D}f(x_0,y_0)\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}. \tag{2}$$

2.5 Properties of the derivative

In sec 2.3, we learned the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is denoted by

$$\mathbf{D}f(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, \quad m \times n \text{ matrix}$$

we call **the matrix of partial derivatives of f at x_0**.

Now we want to know how it interacts with operators, like scalar multiple, sum, product, $\cdots$.

- Constant multiple Rule and Sum Rule:
  If $f : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable at x_0, then we have

  (1) $\mathbf{D}(cf)(x_0) = c\mathbf{D}f(x_0)$

  (2) $\mathbf{D}(f+g)(x_0) = \mathbf{D}f(x_0) + \mathbf{D}g(x_0)$

- Product Rule and Quotient Rule:
  If $f : \mathbb{R}^n \to \mathbb{R}^1$, $g : \mathbb{R}^n \to \mathbb{R}^1$ are differentiable at x_0, then

  (1) $(fg)(x) = f(x)g(x)$ is differentiable at x_0 and

  $$\mathbf{D}(fg)(x_0) = g(x_0)\underbrace{\mathbf{D}f(x_0)}_{[\ \ ]_{1\times n}} + f(x_0)\underbrace{\mathbf{D}g(x_0)}_{[\ \ ]_{1\times n}} \qquad [\text{product}]$$

  (2) $\frac{f(x)}{g(x)}$ is differentiable at x_0(Assume that g is never zero) and

  $$\mathbf{D}\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)\mathbf{D}f(x_0) - f(x_0)\mathbf{D}g(x_0)}{[g(x_0)]^2} \qquad [\text{Quotient}]$$

<u>EX</u>: $f(x,y) = xy^2$, $g(t) = (9, 2t)$. Then

$$(f \circ g)(t) = f(g(t)) \overset{\substack{x=9 \\ y=2t}}{=} 9(2t)^2 = 36t^2.$$

- Chain Rule: We consider functions $f : \mathbb{R}^m \to \mathbb{R}^p$, $g : \mathbb{R}^n \to \mathbb{R}^m$. If the domain of f includes the range of g, so $f \circ g$ is defined. Suppose that g is differentiable at x_0, and f is differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at x_0 and

$$\mathbf{D}(f \circ g)(x_0) = \underbrace{\mathbf{D}f(g(x_0))}_{(p\times m)}\ \underbrace{\mathbf{D}g(x_0)}_{(m\times n)} \text{ matrix}$$

Example 1. *Let* $f(u,v) = (u+v, u, v^2)$, $g(x,y) = (x^2+1, y^2)$. *Find* $\boldsymbol{D}(f \circ g)(1,1)$.

3x2 matrix

$f : \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad g : \mathbb{R}^2 \to \mathbb{R}^2$

Df : 3×2 matrix. $\qquad Dg$ (2×2) matrix

$$D(f\circ g)(x,y) = Df(g(x,y))\, Dg(x,y)$$

$$Dg(x,y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}_{2\times 2}.$$

$$Df(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix}_{3\times 2}$$

Find $D(f\circ g)(1,1) = Df(g(1,1))\, Dg(1,1)$.

$$Dg(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$(u, v) = g(1,1) = (2, 1)$.

$$Df(g(1,1)) = Df(\overset{u}{2}, \overset{v}{1}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$D(f\circ g)(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}_{3\times 2}. \#$$

Example 2. *Let* $f(x,y,z) = x^2 + xe^{xy^2z}$, *and* $g(x,y,z) = z^2+1$. *Find* $\boldsymbol{D}(fg)(x,y,z)$.

$$D(fg)(x,y,z)$$
$$= g\,Df(x,y,z) + f\,Dg(x,y,z).$$
$$= (z^2+1)\begin{bmatrix} f_x & f_y & f_z \end{bmatrix} + (x^2 + xe^{xy^2z})\begin{bmatrix} g_x & g_y & g_z \end{bmatrix}.$$
$$= (z^2+1)\begin{bmatrix} 2x + e^{xy^2z} + xy^2z\,e^{xy^2z} & 2y\,x^2z\,e^{xy^2z} & x^2y^2e^{xy^2z} \end{bmatrix}$$
$$+ (x^2 + xe^{xy^2z})\begin{bmatrix} 0 & 0 & 2z \end{bmatrix}$$

Simplify
$$= \begin{bmatrix} (z^2+1)(2x + e^{xy^2z} + xy^2z\,e^{xy^2z}) & (z^2+1)(2yx^2z\,e^{xy^2z}) & \_\_\_\_\_\_ \end{bmatrix}$$

Example 3. *Suppose* $c(t) : \mathbb{R}^1 \to \mathbb{R}^3$ *is a path and* $f : \mathbb{R}^3 \to \mathbb{R}^1$. *Let* $h(t) = f(c(t)) = f(x(t), y(t), z(t))$, *where* $c(t) = (x(t), y(t), z(t))$. *Then*

$$\boldsymbol{D}h(t) = D(f \circ c)(t)$$
$$= Df(c(t))\; Dc(t).$$
$$= \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}_{(x,y,z)=c(t)} \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}.$$
$$= f_x(c(t))\,x'(t) + f_y(c(t))\,y'(t) + f_z(c(t))\,z'(t).$$
$$= \nabla f(c(t)) \cdot c'(t).$$

Here
$$\nabla f = \langle f_x, f_y, f_z \rangle$$
$$c'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

2.4 Introduction to parametrized curves

A function $c(t) : [a, b] \to \mathbb{R}^m$ (that is, $c(t)$ maps from $[a, b]$ to $\mathbb{R}^m$) is a **path** in $\mathbb{R}^m$.

For example,

$x \quad y \quad \Rightarrow \quad y = x^2.$

$$c(t) = (t, t^2) \text{ for } t \text{ in } [-1, 1].$$

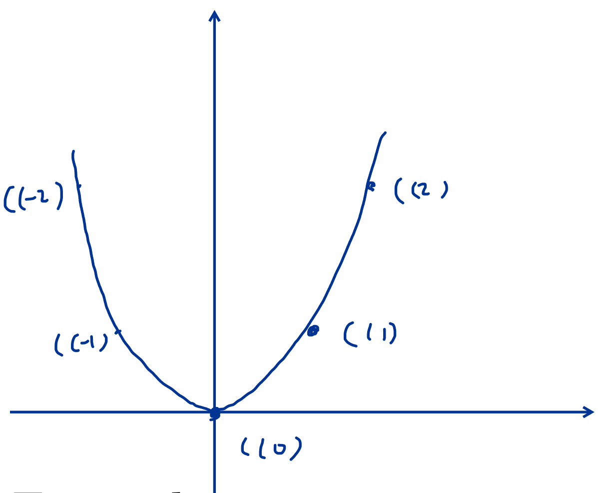
Here we refer to t as a free parameter.

$c(0) = (0, 0)$

$c(1) = (1, 1)$

$c(-1) = (-1, 1)$

$c(2) = (2, 4).$



We call $c(t)$ parametrize the parabola $y = x^2$.

or call $c(t)$ a parametrization of $y = x^2$

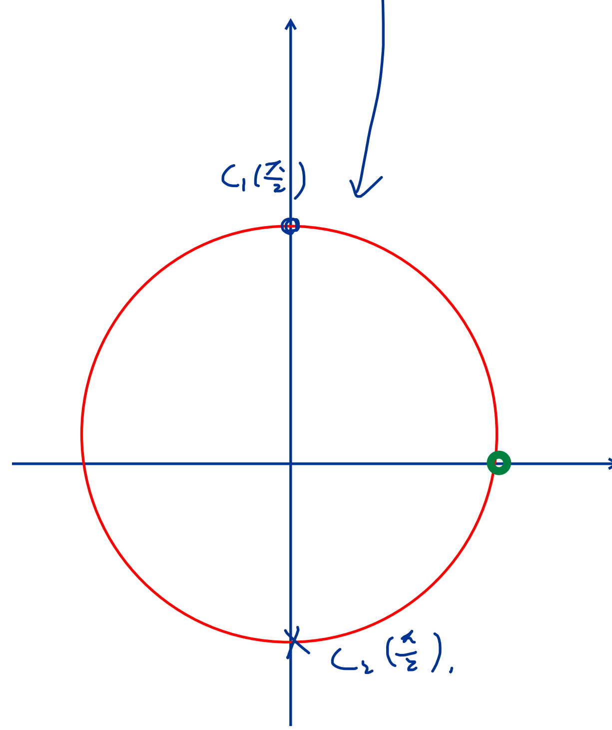
Example 1.

$$c_1(t) = (\cos(t), \sin(t)), \ 0 \le t \le 2\pi, \qquad x^2 + y^2 = 1.$$

$$c_2(t) = (\cos(3t), \sin(3t)), \ 0 \le t \le 2\pi/3$$

Ex: $t = \pi/2$, $c_1(\frac{\pi}{2}) = (0, 1)$

$c_2(\frac{\pi}{2}) = (0, -1)$



Derivatives of parametrized curves

Consider the function $c : \mathbb{R}^1 \to \mathbb{R}^m$ where

$$c(t) = (c_1(t), c_2(t), \cdots, c_m(t)).$$

The **Derivatives** of the function $c : \mathbb{R}^1 \to \mathbb{R}^m$ at time t is

$$\mathbf{D}c(t) = \begin{bmatrix} c_1'(t) \\ \vdots \\ c_m'(t) \end{bmatrix}_{m\times 1}$$

matrix of partial derivative

$$c'(t) = \langle c_1'(t), c_2'(t), \ldots, c_m'(t) \rangle.$$

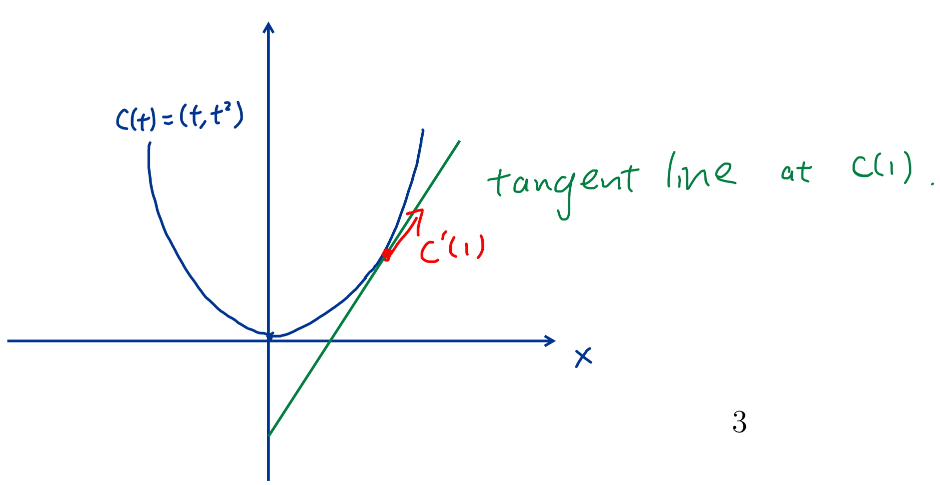
We could also define the derivatives by using the limit definition:

$$c'(t) = \lim_{h\to 0} \frac{c(t+h) - c(t)}{h}$$

- If $c'(t) \neq 0$, the velocity $c'(t)$ gives a "direction" for the tangent line to the curve at t. We call $c'(t)$ is a "tangent vector" to the curve at t.
- Note that $c'(t)$ is the velocity of the curve $c(t)$. The **speed** of the path is $\|c'(t)\|$.
- Tangent line to a path at point $c(t_0)$ is

$$l(t) = c(t_0) + (t - t_0)c'(t_0).$$

EX: $c(t) = (t, t^2)$



Example 2. *Let* $c(t) = (\overset{x}{t^3}, \overset{y}{e^t}, \overset{z}{2t})$ *and* $f(x, y, z) = (x^2 - y^2, 2y, z^2)$.

1. *Find* $(f \circ c)(1)$.
2. *Find the equation of the tangent line to the curve* $f \circ c$ *at* $t = 1$.

1. $(f \circ c)(1) = f(c(1))$

$$= \langle (t^3)^2 - (e^t)^2, 2e^t, (2t)^2 \rangle$$

$$= \langle t^6 - e^{2t}, 2e^t, 4t^2 \rangle.$$

2. Tangent line

$$l(t) = (f \circ c)(1) + (t-1)(f \circ c)'(1)$$

$$= (1 - e^2, 2e^1, 4) + (t-1)(6 - 2e^2, 2e, 8) \quad \#$$

Compute $(f \circ c)'(1)$.

$$(f \circ c)'(t) = \langle 6t^5 - 2e^{2t}, 2e^t, 8t \rangle.$$

$$(f \circ c)'(1) = \langle 6 - 2e^2, 2e^1, 8 \rangle.$$

A vector-valued function $c : \mathbb{R} \to \mathbb{R}^m$ can be viewed as a parametrization of a curve. We think of t as time and the parametrization $c(t)$ as being the **position** of an object at time t. The curve parametrized by $c(t)$ is the **trajectory** of this object.

Go back to Example 1,

$$c_1(t) = (\cos(t), \sin(t)),\ 0 \le t \le 2\pi,$$
$$c_2(t) = (\cos(3t), \sin(3t)),\ 0 \le t \le 2\pi/3.$$

So c_1 and c_2 are parametrization of an unit circle.

Velocity:

$$c_1'(t) = (-\sin(t),\ \cos(t))$$
$$c_2'(t) = (-3\sin(3t),\ 3\cos(3t))$$

Speed:

$$\|c_1'(t)\| = \|(-\sin(t), \cos(t))\| = 1$$
$$\|c_2'(t)\| = \|(-3\sin(3t), 3\cos(3t))\| = 3$$

Thus, $c_2(t)$ is 3 times faster than $c_1(t)$.