Quick Reivew from last week

- The velocity of the path $c(t)$ is $c'(t)$. The speed of the path is $\|c'(t)\|$.
- Tangent line to a path at point $c(t_0)$ is

$$l(t) = c(t_0) + (t - t_0)c'(t_0).$$

- $\mathbf{D}(fg)(x_0) = g(x_0)\mathbf{D}f(x_0) + f(x_0)\mathbf{D}g(x_0)$
- $\mathbf{D}\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)\mathbf{D}f(x_0) - f(x_0)\mathbf{D}g(x_0)}{[g(x_0)]^2}$
- $\mathbf{D}(f \circ g)(x_0) = \mathbf{D}f(g(x_0))\ \mathbf{D}g(x_0)$
  - Example:
    Suppose $c(t) = (x(t), y(t), z(t))$ is a path and $f : \mathbb{R}^3 \to \mathbb{R}^1$. Then

$$\mathbf{D}(f \circ c)(t) = \mathbf{D}f(c(t))\mathbf{D}c(t)$$

    or

$$\mathbf{D}(f \circ c)(t) = \nabla f(c(t)) \cdot c'(t)$$

- Directional Derivatives $\mathbf{D}_{\mathbf{v}}f(x) = \nabla f(x) \cdot \underline{\mathbf{v}}$ — unit vector
- If $\nabla f(x) \neq 0$, then $\nabla f(x)$ points in the direction of the fastest increase of f.
- The tangent plane of the level surface $f(x, y, z) = c$ (c is constant) at point (x_0, y_0, z_0) is

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

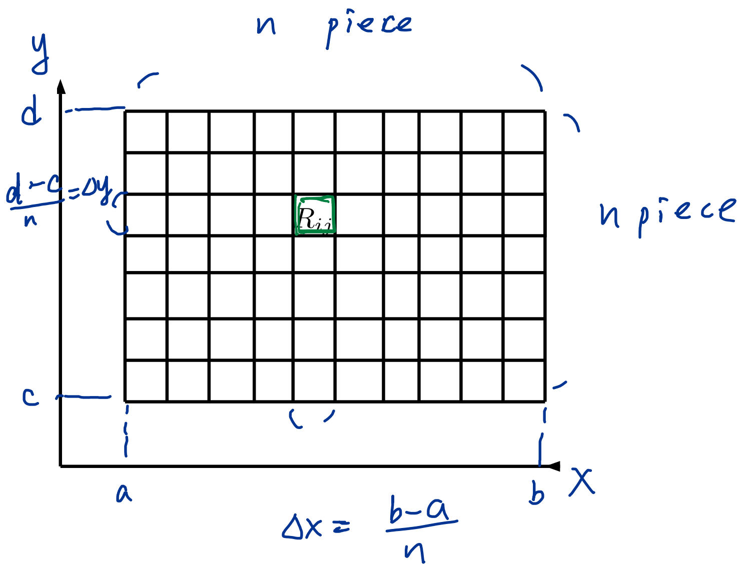
  if $\nabla f(x_0, y_0, z_0) \neq 0$.

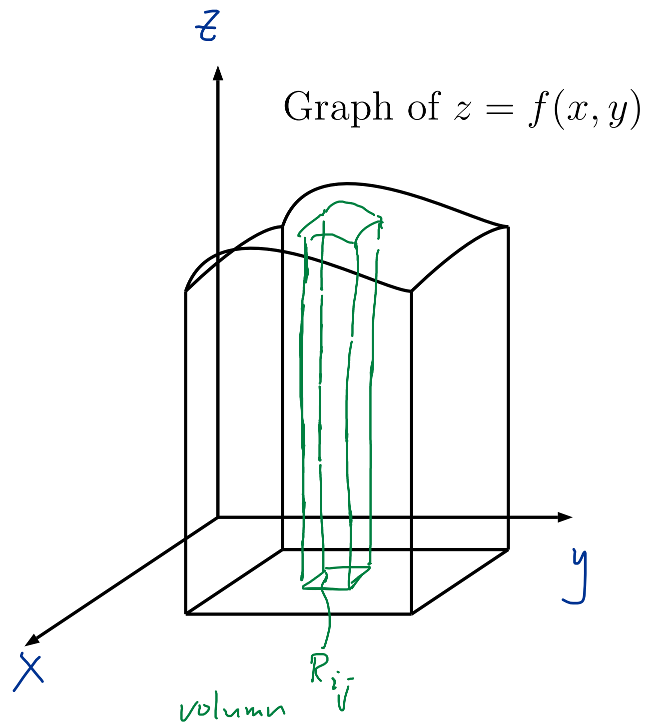
Quiz 3 : 5.1. – 5.3.

5.1-5.2 Double integrals

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ ($f(x,y)$) defined on a rectangle $R = [a,b] \times [c,d]$ (that is, $a \le x \le b, c \le y \le d$).

Divide $[a,b]$ and $[c,d]$ into n equal subintervals.





area of R_{ij} = $\Delta x \Delta y = \left(\frac{b-a}{n}\right)\left(\frac{d-c}{n}\right)$.

volumn = area(R_{ij}) $\cdot f(x_{ij}, y_{ij})$.

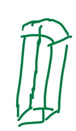
Here (x_{ij}, y_{ij}) is in R_{ij}.

There are n^2 subrectangles R_{ij} in R. We define

$$\int\int_R f(x,y)dA$$

$$= \iint_R f(x,y)\, dx\, dy$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} \underbrace{f(x_{ij}, y_{ij})}_{\text{height of}} \overbrace{\Delta x \Delta y}^{\text{area of } R_{ij}} \quad \text{if the limit exists.}$$



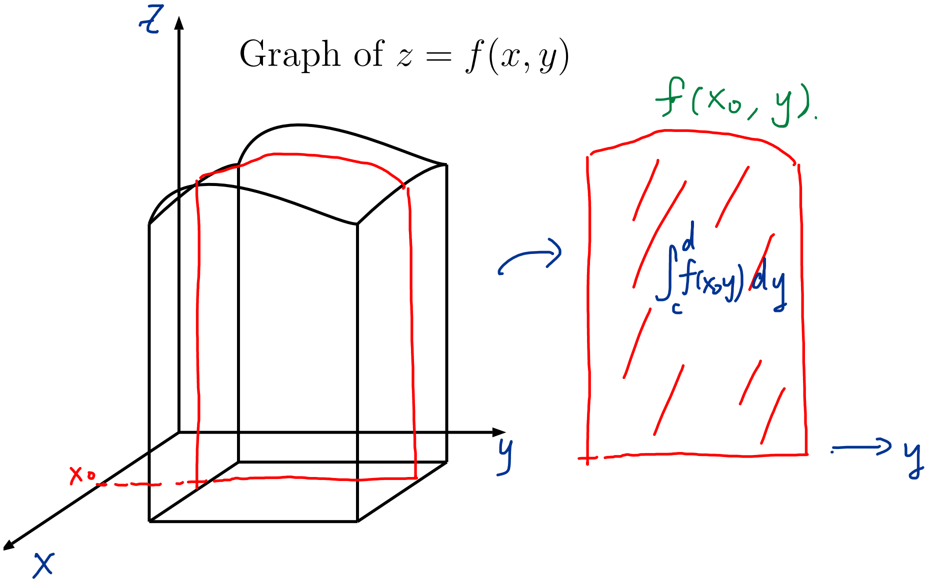
NOTE that if $f \ge 0$,

then $\iint_R f\, dA$ is the volume of the 3D region under the graph of f ($z = f(x,y)$).

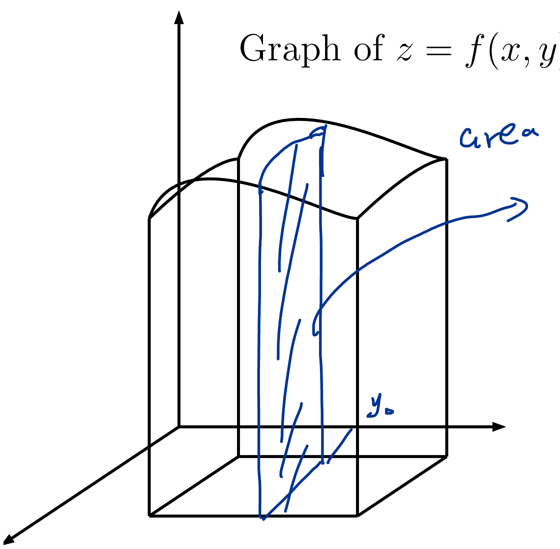
§How do we compute double integral of $f(x, y)$ over the domain R, that is, $\int\int_R f dA$?

Recall that R is a rectangle with $a \le x \le b, c \le y \le d$.

Ans: By "Slice principle",



$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$



$$\int_a^b f(x, y_0)\,dx$$

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y_0)\,dx\right] dy$$

We can switch between these two orders as stated below.

[Fubini Theorem]

Fact. *If f is continuous on $R = [a, b] \times [c, d]$, then*

$$\int\int_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

Example 1. *Evaluate the integral*

$$\int\int_R xye^{-x^2+y^2}dA,$$

where $R = \overset{x}{[0,1]} \times \overset{y}{[0,2]}$.

Take $y=1$

$\int x\, e^{-x^2+1}\, dx$

$$\int_0^2 \left[\int_0^1 xy\, e^{-x^2+y^2}\, dx\right] dy$$

$u = -x^2+y^2$.

$du = -2x\, dx$.

$$= \int_0^2 \left[\int_{y^2}^{-1+y^2} \underbrace{-\tfrac{1}{2}\, y\, e^u}\, du\right] dy$$

$$= \int_0^2 \left(-\tfrac{1}{2} y\, e^u \Big|_{y^2}^{-1+y^2} \right) dy$$

$$= \int_0^2 -\tfrac{1}{2}\, y\, \underbrace{e^{-1+y^2}} + \underbrace{\tfrac{1}{2} y\, e^{y^2}}\, dy$$

$$= -\tfrac{1}{4}\, e^{-1+y^2} + \tfrac{1}{4} e^{y^2} \Big|_0^2$$

$$= \tfrac{1}{4} e^4 - \tfrac{1}{4} e^3 + \tfrac{1}{4} e^{-1} - \tfrac{1}{4}. \ \#$$

Remark: If you compute $\int_0^1 \int_0^2 xy\, e^{-x^2+y^2}\, dy\, dx$, by Fubini's thm., you should have the same answer. #

observe ① $\int 2e^{2x} dx$.

② $\int y e^{2y} dy$.

Example 2. *Evaluate the integral*

$$\int\int_R ye^{xy}dA,$$

where $R = [1,5] \times [0,2]$.

$$\begin{cases} ① & \iint y e^{xy}\, dx\, dy \text{ (easier)} \\ ② & \iint y e^{xy}\, dy\, dx \end{cases}$$

① $\int_0^2 \int_1^5 y e^{xy}\, dx\, dy$

$= \int_0^2 \left(e^{xy} \Big|_1^5 \right) dy$

$= \int_0^2 \left(e^{5y} - e^{y} \right) dy$

$= \frac{1}{5} e^{5y} - e^{y} \Big|_0^2$

$= \frac{1}{5} e^{10} - e^{2} + \frac{4}{5}.$

I. by P.:

$\int u\, dv = uv - \int v\, du.$

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② $\int_1^5 \left[\int_0^2 \underbrace{y}_{u}\, \underbrace{e^{xy}\, dy}_{dv} \right] dx.$

$u = y \qquad dv = e^{xy} dy$

$du = dy \qquad v = \frac{1}{x} e^{xy}.$

$= \int_1^5 \left[y \frac{1}{x} e^{xy} \Big|_0^2 - \int_0^2 \frac{1}{x} e^{xy}\, dy \right] dx$

$= \int_1^5 \left[\frac{2}{x} e^{2x} - \int_0^2 \frac{1}{x} e^{xy}\, dy \right] dx$

$= \int_1^5 \left(\frac{2}{x} e^{2x} \right) - \frac{1}{x^2} e^{xy} \Big|_0^2\, dx$

$= \underbrace{\int_1^5 \frac{2}{x} e^{2x}}_{\text{I. by P.}} - \frac{1}{x^2} e^{2} + \frac{1}{x^2}\, dx,$

$=$ the same answer.