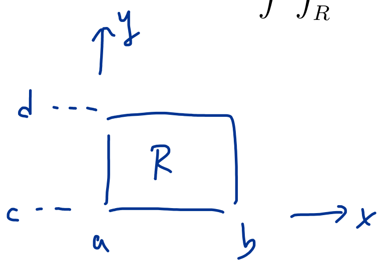
Math 2374 Spring 2018 - Week 5

Quick Reivew from previous lecture

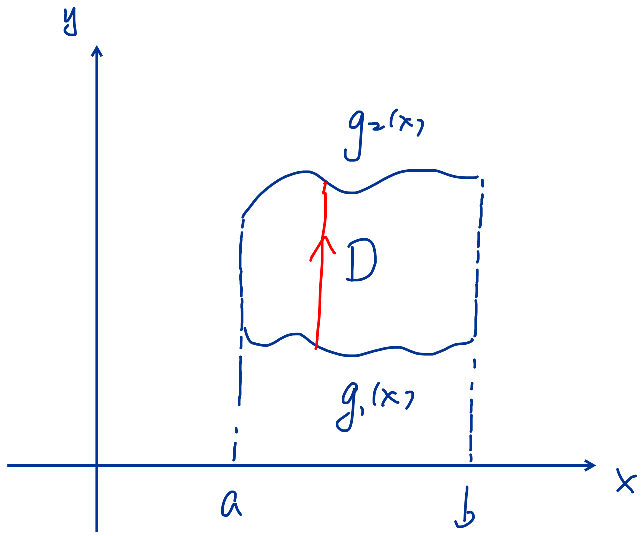
Fact. *(Fubini's Theorem) If f is continuous on $R = [a,b] \times [c,d]$, then*

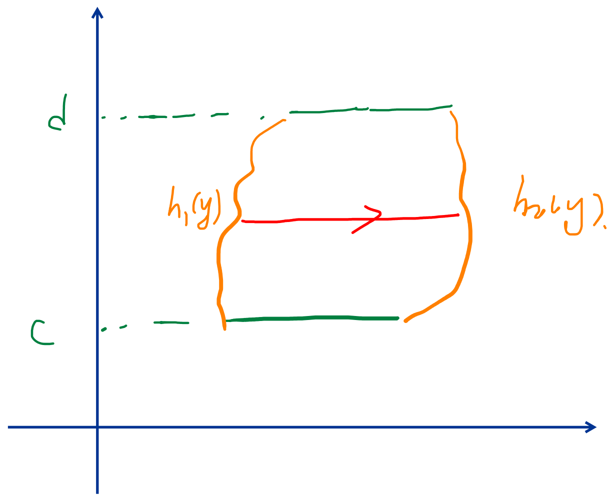
$$\int\int_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$



5.3 Double Integralas over General Regions

In this section, we want to set up the double integral of $f(x, y)$ over regions D:





(1) Bounded by 2 functions of x:

$$D: \quad a \leq x \leq b$$
$$g_1(x) \leq y \leq g_2(x)$$

We call the region D "y-simple."

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx$$

(2) Bounded by 2 functions of y:

$$D: \ h_1(y) \leq x \leq h_2(y)$$
$$c \leq y \leq d$$

We call the region D "x-simple."

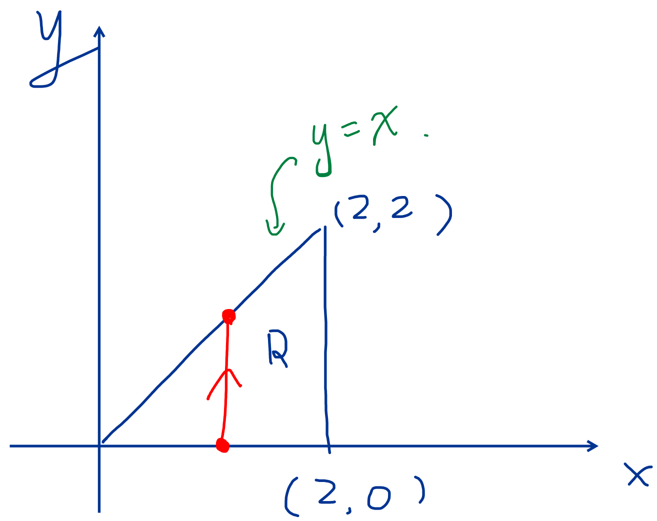
$$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$$

Note that a region D is both x-simple and y-simple, we call D is "simple".

Example 1. *Evaluate the integral*

$$\int\int_R (x^3y + e^x)dA,$$

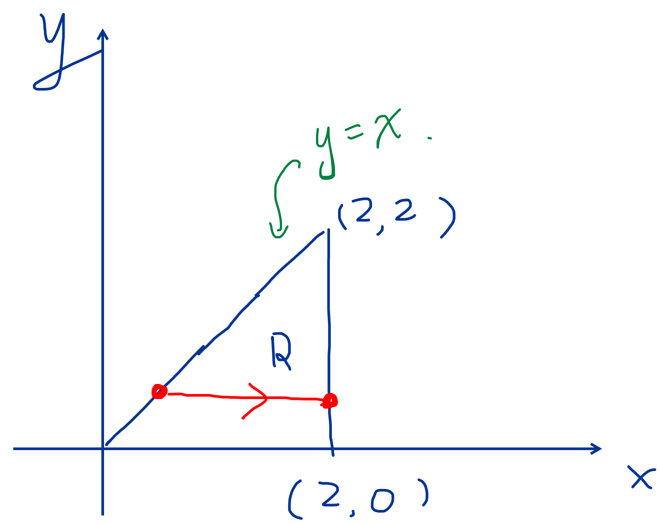
where R is the triangle with vertices $(0,0)$, $(2,0)$, $(2,2)$.



(y - simple)

$0 \le x \le 2$

$0 \le y \le x.$



(x-simple)

$y \le x \le 2$

$0 \le y \le 2$

$$\int_0^2 \int_0^x (x^3y + e^x)\,dy\,dx.$$

$$= \int_0^2 x^3 \frac{y^2}{2} + e^x y \Big|_0^x \,dx$$

$$= \int_0^2 \left(\frac{x^5}{2} + xe^x\right) dx$$

$$= \frac{1}{12}x^6 \Big|_0^2 + \int_0^2 x e^x \,dx$$

$$= \frac{1}{12}x^6 \Big|_0^2 + xe^x - e^x \Big|_0^2$$

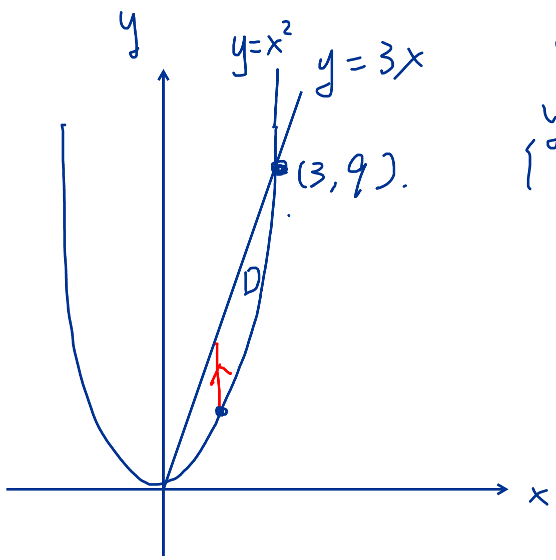
$$= \frac{16}{3}\cancel{\frac{32}{6}}\cancel{\frac{64}{12}} + 2e^2 - e^2 + 1.$$

$$\boxed{\int_0^2 \int_y^2 (x^3y + e^x)\,dx\,dy = \sim \text{ exercise!}}$$

I. by P.

$\int u\,dv = uv - \int v\,du.$

Example 2. *Integrate $f(x, y) = xy^2$ over the region D which is bounded by $y = 3x$ and $y = x^2$.*



Find intersection

$$\begin{cases} y = 3x \\ y = x^2 \end{cases}$$

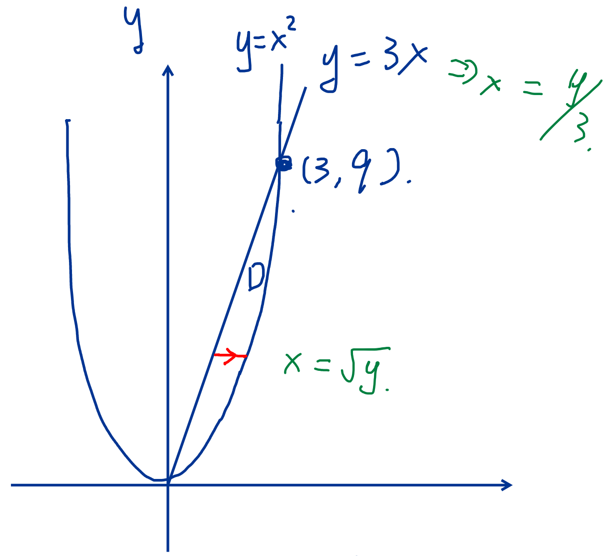
$3x = x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0, 3$

$(0,0)$ $(3,9)$.



[y - simple]

$0 \le x \le 3$

$x^2 \le y \le 3x$.

$$\int_0^3 \left(\int_{x^2}^{3x} xy^2 \, dy \right) dx$$

$$= \int_0^3 \left(x \frac{y^3}{3} \Big|_{x^2}^{3x} \right) dx$$

$$= \frac{9^4}{40}.$$

[x - simple].

$\frac{y}{3} \le x \le \sqrt{y}$.

$0 \le y \le 9$

$$\int_0^9 \left(\int_{y/3}^{\sqrt{y}} xy^2 \, dx \right) dy$$

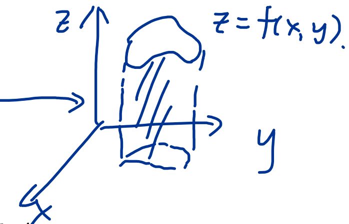
$$= \frac{9^4}{40}.$$

Example: Find area of region D.

$$\iint_D 1 \, dA = \int_0^3 \int_{x^2}^{3x} 1 \, dy \, dx = \int_0^3 (3x - x^2) \, dx.$$

§Double integrals as volume

volume $= \iint_D f \, dA$

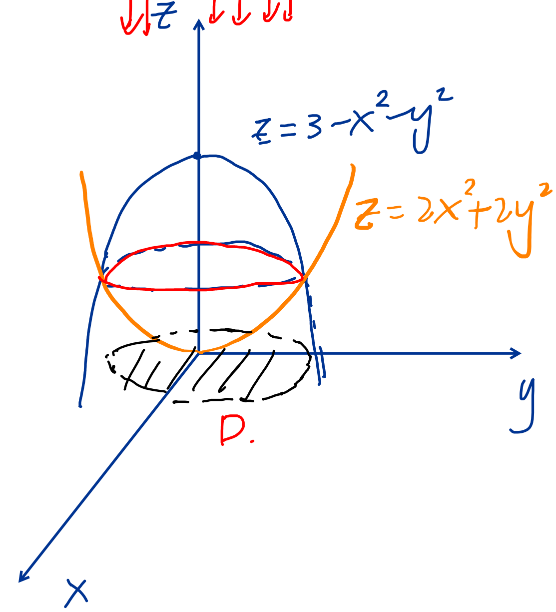


If $f(x,y) > g(x,y)$, then the double integral

$$\int\int_D (f(x,y) - g(x,y))dA$$

is the "volume" between the surface $z = f(x,y)$ and the surface $z = g(x,y)$.

Example 3. *Calculate the volume between the surfaces $z = f(x,y) = 3 - x^2 - y^2$ and the surface $z = g(x,y) = 2x^2 + 2y^2$. (Example 2 in Math insight)*



Volume

$$= \iint_D (3 - x^2 - y^2) - (2x^2 + 2y^2) \, dA.$$

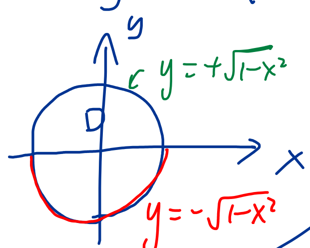
Find interseciton of $z = 3 - x^2 - y^2$, $z = 2x^2 + 2y^2$.

$$3 - x^2 - y^2 = 2x^2 + 2y^2.$$

$$3x^2 + 3y^2 = 3$$

$$\boxed{x^2 + y^2 = 1}$$

D: unit disk $x^2 + y^2 \le 1$.



$$y = \pm\sqrt{1 - x^2}$$

method 1:

$$-1 \le x \le 1$$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3 - 3x^2 - 3y^2) \, dy \, dx$$

$= \sim$

§1.4.

method 2: (polar coord.)

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

D: $0 \le r \le 1$

$0 \le \theta \le 2\pi$.

$$\int_0^{2\pi} \int_0^1 (3 - 3r^2) \, r \, dr \, d\theta. \quad r^2 = x^2 + y^2$$

$$= 3\pi/2.$$

5.4 Changing the order of Integration

Sometimes, by changing the order of the iterated integral, we will have a much easier integral to solve.

See EX2 in NOTE 5.1, 5.2.

$$\int_1^5 \int_0^2 ye^{xy} dy dx$$

$$\int_0^2 \int_1^5 ye^{xy} dx dy$$

Example 4. *Integrate*

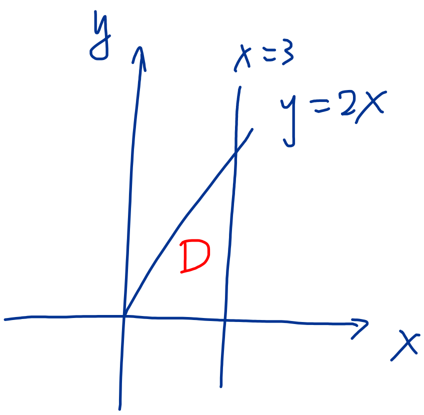
$$\int_0^6 \int_{y/2}^3 e^{x^2} dx dy$$

[x-simple]

$y/2 \le x \le 3$ $\Rightarrow$ $x = y/2$

$x = 3$.

$0 \le y \le 6$

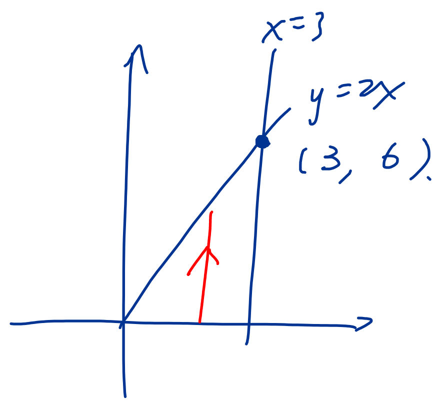


$\Longrightarrow$

[y-simple]

$0 \le x \le 3$

$0 \le y \le \boxed{2x}$



$$\int_0^3 \int_0^{2x} e^{x^2} \, dy \, dx$$

$$= \int_0^3 e^{x^2} (2x) \, dx$$

$$= e^{x^2} \Big|_0^3 = e^9 - 1. \#$$