Math 2374 Spring 2018 - Week 6

Quick Review from previous lecture

(1)

Fact. *(Fubini's Theorem) If f is continuous on $R = [a, b] \times [c, d]$, then*

$$\int\int_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

$R = [a,b] \times [c,d]$.

(2) y-simple:

$$\int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)dy}dx.$$

$\Rightarrow$ $a \le x \le b$, $g_1(x) \le y \le g_2(x)$

x-simple:

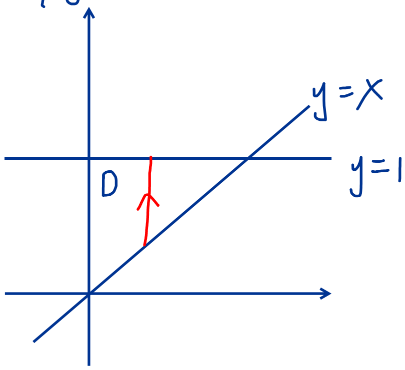
$$\int_c^d \underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)dx}dy.$$

Quiz 4: 5.4 - 5.5.

Example 5. *Integrate*

$$\underbrace{\int_0^1}\underline{\int_x^1} \sin(y^2)dydx$$

[y-simple]

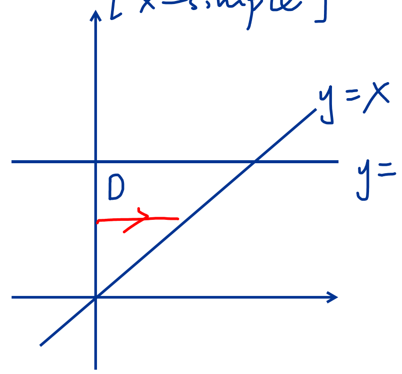


$$\begin{cases} 0 \le x \le 1 \\ x \le \underline{y \le 1} \end{cases}$$

$$\begin{cases} x = y \\ y = 1. \end{cases}$$

$$\underline{\underline{dydx \to dxdy}} \Longrightarrow$$

[x-simple]



$0 \le x \le y$

$0 \le y \le 1.$

$$\int_0^1\int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 y\sin(y^2)\, dy$$

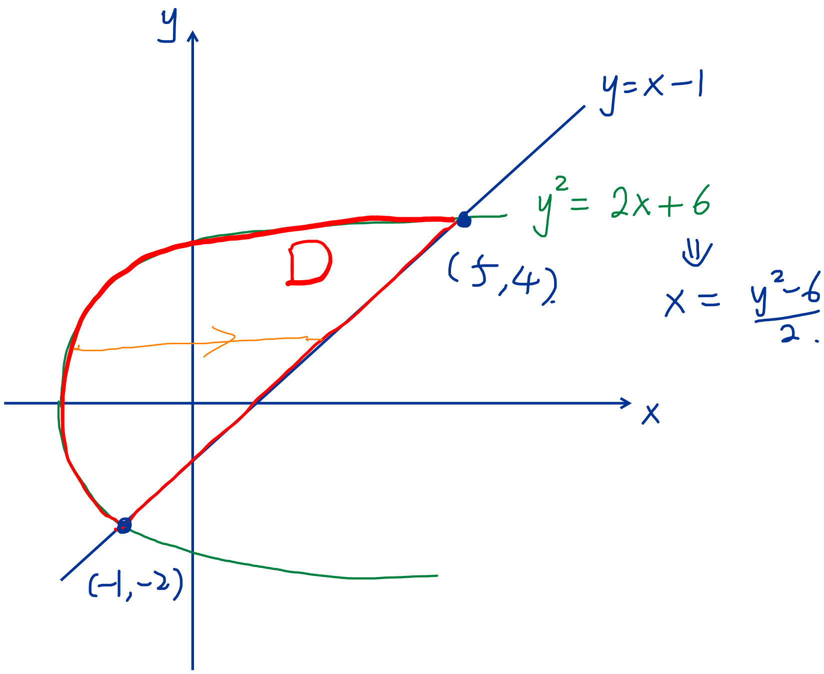
$$= -\frac{1}{2}\cos(y^2)\Big|_0^1$$

$$= -\frac{1}{2}\cos 1 + \frac{1}{2}. \#$$

Example 6. *Integrate*

$$\iint_D xy\,dA,$$

where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.



Find intersections.

$x = y+1$

$x = \frac{y^2-6}{2}$

$y+1 = \frac{y^2-6}{2}$.

$2y+2 = y^2-6$

$y^2-2y-8=0$.

$(y-4)(y+2)=0$

$y = 4, -2$.

$(5,4)$, $(-1,-2)$

[x− simple]

$\frac{y^2-6}{2} \le x \le y+1$

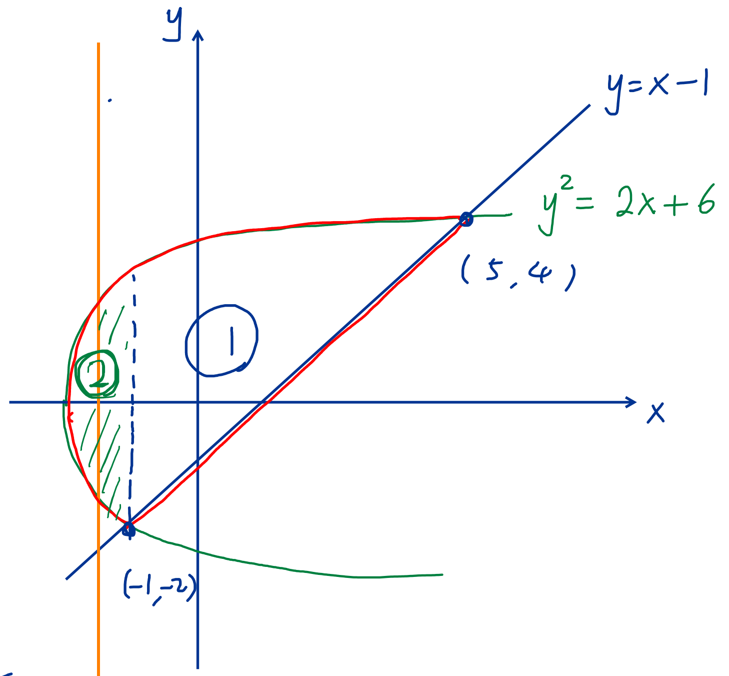
$-2 \le y \le 4$.

$$\int_{-2}^{4}\int_{\frac{y^2-6}{2}}^{y+1} xy \, dx\, dy.$$

$$= \int_{-2}^{4} \frac{x^2}{2} y \Big|_{\frac{y^2-6}{2}}^{y+1} dy$$

$$= \int_{-2}^{4} \frac{y}{2}\left((y+1)^2 - \left(\frac{y^2-6}{2}\right)^2\right) dy$$

$$= \underline{36}.$$



$\Rightarrow y = \pm\sqrt{2x+6}.$

[y - simple]

① $-1 \le x \le 5$

$x-1 \le y \le \sqrt{2x+6}$

② $-3 \le x \le -1$

$-\sqrt{2x+6} \le y \le \sqrt{2x+6}.$

$$\int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy \, dy\, dx + \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy \, dy\, dx.$$

= the same solution as previous page.

5.5 The triple Integral

Recall: in 5.1, 5.2

We consider a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined on a rectangle $R = [a, b] \times [c, d]$ (that is, $a \leq x \leq b, c \leq y \leq d$).

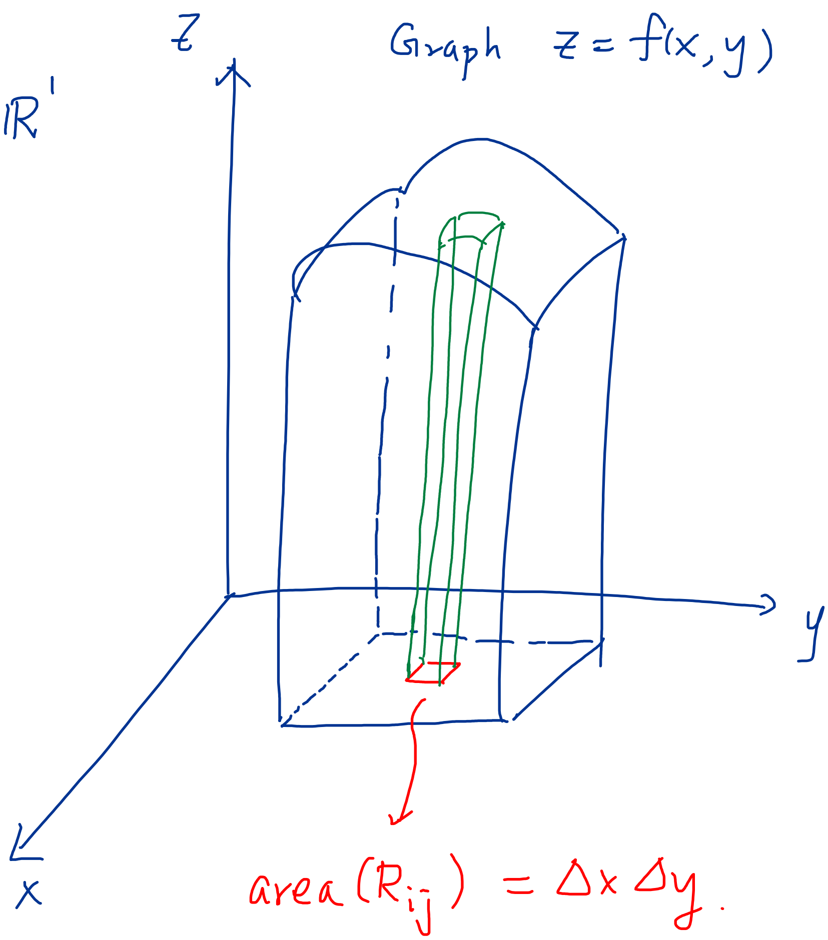
Divide $[a, b]$ and $[c, d]$ into n equal subintervals.

$$\Delta x = \frac{b-a}{n}, \qquad \Delta y = \frac{d-c}{n}$$

There are n^2 subrectangles R_{ij} in R. We define

$$\int\int_R f(x,y)dA = \int\int_R f(x,y)dxdy$$
$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_{ij}, y_{ij})\Delta x \Delta y \text{ if the limit exists.}$$

$f : \mathbb{R}^2 \to \mathbb{R}^1$



Now in 5.5, the triple integral, we want to consider $f : \boxed{\mathbb{R}^3} \to \mathbb{R}^1$ defined on a rectangle $W = [a,b] \times [c,d] \times [s,t]$.

Divide $[a,b]$ and $[c,d]$ and $[s,t]$ into n equal subintervals.

$$\Delta x = \frac{b-a}{n}, \qquad \Delta y = \frac{d-c}{n}, \qquad \Delta z = \frac{t-s}{n}$$

There are n^3 subrectangles W_{ijk} in W. We define

$$\int\int\int_W f(x,y,z)dV = \lim_{n\to\infty} \sum_{i,j,k=1}^{n} f(x_i, y_j, z_k)\Delta x \Delta y \Delta z \text{ if the limit exists.}$$

§How do we compute $\int\int\int_W f(x,y,z)dV$ with $W = \overset{x}{[a,b]} \times \overset{y}{[c,d]} \times \overset{z}{[s,t]}$?

Ans:

Reduction to **Iterated Integrals**.

$$\begin{aligned}\int\int\int_W f(x,y,z)dV &= \int_s^t \int_c^d \int_a^b f(x,y,z)dxdydz \\ &= \int_s^t \int_a^b \int_c^d f(x,y,z)dydxdz \\ &= \int_a^b \int_c^d \int_s^t f(x,y,z)dzdydx \\ &= \cdots\end{aligned}$$

* The three integrals (that is, $\int dx$, $\int dy$, $\int dz$) can be arranged in any of 6 orders.

Example 1. *Evaluate the integral*

$$\int\int\int_W xyz^2 dV,$$

where $W = [\overset{x}{0,3}] \times [\overset{y}{2},4] \times [\overset{z}{0},1]$.

$$\int_0^1\int_2^4\int_0^3 x\,y\,z^2\,dx\,dy\,dz$$

$$= \int_0^1\int_2^4 \left(\frac{x^2}{2}\,yz^2 \Big|_0^3\right) dy\,dz$$

$$= \int_0^1\int_2^4 \frac{9}{2}\,y\,z^2\,dy\,dz$$

$$= \int_0^1 \left(\frac{9}{4}\,y^2 z^2 \Big|_2^4\right) dz$$

$$= \int_0^1 \frac{9}{\cancel{4}}(\overset{3}{\cancel{16-4}})\,z^2\,dz$$

$$= \int_0^1 27\,z^2\,dz$$

$$= 9. \quad \#$$

<u>Remark</u>:

you will get the same solution by doing different order, like

$$\iiint \cdots dy\,dx\,dz$$

$$\iiint \cdots dy\,dz\,dx$$

$\vdots$

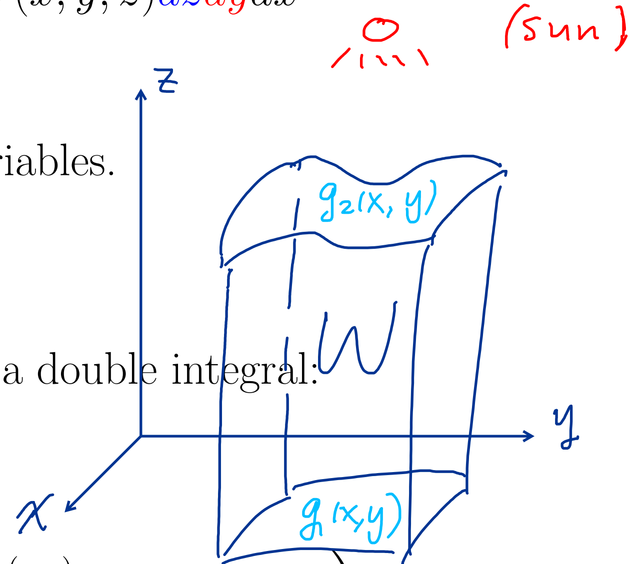
§Triple integrals over "elementary regions"

A region W is called an **elementary region** if it can be described in the form

$$a \ \leq \ x \ \leq \ b$$
$$h_1(x) \ \leq \ y \ \leq \ h_2(x)$$
$$g_1(x,y)) \ \leq \ z \ \leq \ g_2(x,y)$$

Then

$$\int\int\int_W f dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z) dz dy dx$$



Similarly, for the other orderings of the three variables.

There are 2 ways to reduce a triple integral into a double integral:

- **Shadow method:**
  Imagine a sun is on z axes.

$$\int\int\int_W f(x,y,z) dV = \int\int_{shadow} \left(\int_{bottom(x,y)}^{top(x,y)} f(x,y,z) dz \right) dxdy.$$

  Imagine a sun is on x axes.

$$\int\int\int_W f(x,y,z) dV = \int\int_{shadow} \left(\int_{bottom(y,z)}^{top(y,z)} f(x,y,z) dx \right) dydz.$$

  Imagine a sun is on y axes.

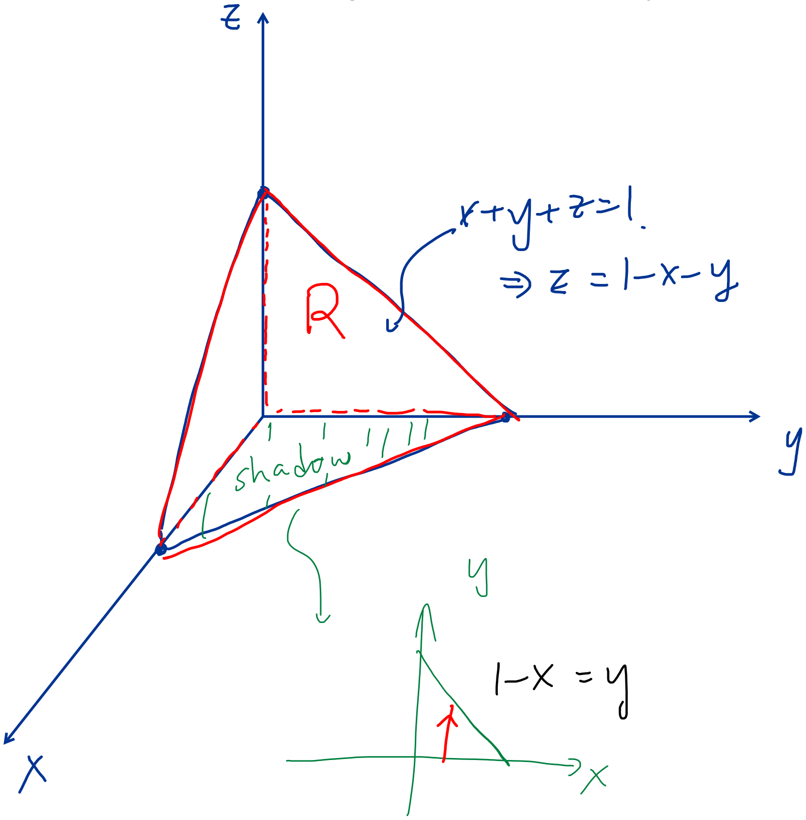
$$\int\int\int_W f(x,y,z) dV = \int\int_{shadow} \left(\int_{bottom(x,z)}^{top(x,z)} f(x,y,z) dy \right) dxdz.$$

- **Cross section method:** See math insight. Sometimes these two methods are interchangeable. We will focus on Shadow method in lecture.

Example 2. *Evaluate*

$$\iiint_R x\,dV$$

with R is the region enclosed by the coordinate planes and $x + y + z = 1$.



$x = 0$ (yz plane)

$y = 0$ (xz plane)

$z = 0$ (xy plane)

shadow method:

shadow region
$$\boxed{\begin{array}{l} 0 \le x \le 1. \\ 0 \le y \le 1-x \end{array}}$$

$$0 \le z \le 1-x-y.$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \quad dz \; dy \; dx$$

$$= 1/24. \quad \#$$

Example 3. *Evaluate*

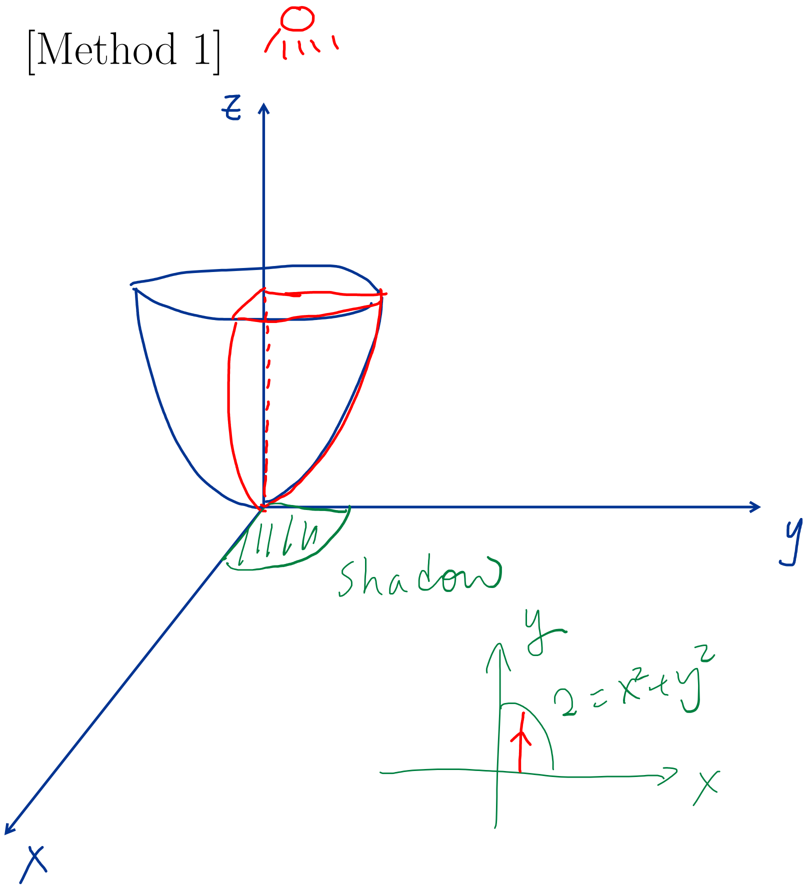
$$\int\int\int_R x dV$$

with R is the region enclosed by the planes $x = 0$, $y = 0$, and $z = 2$ and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0$, $y \geq 0$.

*Note that to plot the paraboloid $z = x^2 + y^2$. One way is using the following code in Mathematica (see Lab 07 file):

```
f[r_, theta_] = {r Cos[theta], r Sin[theta], r^2}
ParametricPlot3D[f[r, theta], {r, 0, 1}, {theta, 0, 2 Pi}]
```

[Method 1]



$0 \leq x \leq \sqrt{2}$

$0 \leq y \leq \sqrt{2-x^2}$

$x^2+y^2 \leq z \leq 2$