Math 2374 Spring 2018 - Week 8

(1) The **line integral** of a scalar-value function f along the curve C parametrized by $c(t),\ a \le t \le b$, is defined to be

$$\int_C f ds = \int_a^b f(c(t))\|c'(t)\|dt.$$

(2) If we let $f(x, y, z)$ denote the mass density at (x, y, z) and suppose the image of $c(t)$ represents a wire, then

$$\int_c f ds = \int_a^b f(c(t))\|c'(t)\|dt$$

can be viewed as the **total mass** of the wire.

(3) The **line integral** of a vector field F along the curve C that is parametrized by $c(t),\ a \le t \le b$, is defined to be

$$\int_C F \cdot d\mathbf{s} = \int_a^b F(c(t)) \cdot c'(t)dt.$$

$$\begin{aligned}\int_c F \cdot d\mathbf{s} &= \int_c Pdx + Qdy + Rdz \\ &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt.\end{aligned}$$

Recall F is force vector field, total work is $\int F \cdot d\vec{s}$

8.1 Green's Theorem

Definition:

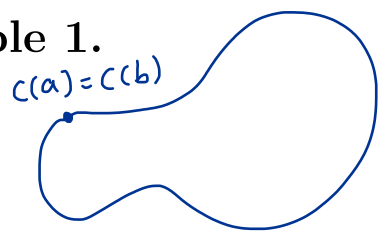
Let the curve C parametrized by $c(t), a \le t \le b$. If

(1) $c(a) = c(b)$. — closed curve

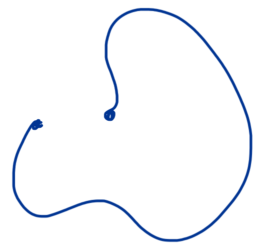
(2) No self intersection. — simple curve

We call C, **a simple closed curve**.

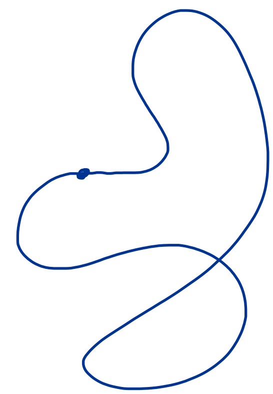
Example 1.



simple, closed



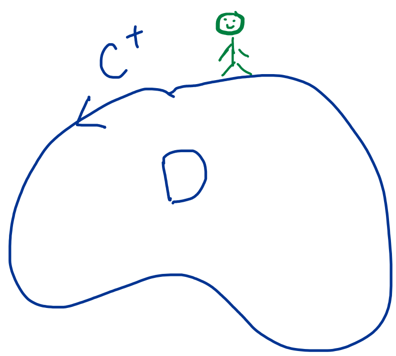
simple,
NOT closed



closed,
NOT simple.

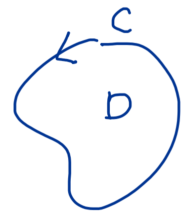
A simple closed curve C has two orientations:

1. **counterclockwise (positive)** orientation, denoted as C^+.
2. **clockwise (negative)** orientation, denoted as C^-.



We call C^+ "the positively oriented boundary of the region D" if you walk along the curve C^+, the region D is always on your "left" hand side.

Fact. *(Green's Theorem)*
Let D be a simple region and let C be the positively oriented boundary of the region D. Suppose $F = (P(x,y), Q(x,y), 0)$. Then



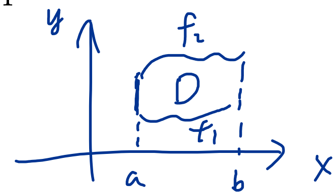
$$\int_C F \cdot d\boldsymbol{s} = \int\int_D (\text{curl} F \cdot \boldsymbol{k}) dA$$

$\text{curl } F = \nabla \times F = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$. $k = (0,0,1)$

D is simple $\quad \text{curl } F \cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

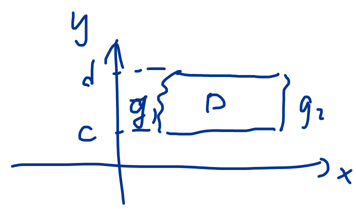
Recall: A simple region means this region can be expressed in both forms:

$$a \leq x \leq b,$$
$$f_1(x) \leq y \leq f_2(x),$$



and

$$g_1(y) \leq x \leq g_2(y),$$
$$c \leq y \leq d.$$



We also use the notation ∂D (which represents the boundary of region D) to denote the positive (counterclockwise) orientation C^+. Thus, we can rewrite Green's theorem as follows:

$$\int_{\partial D} P dx + Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

boundary of D
positively oriented.

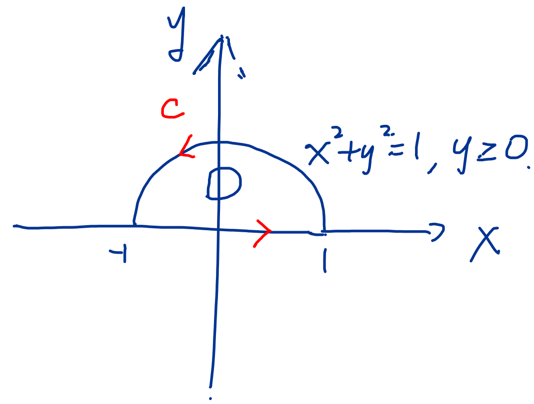
Thus, $\displaystyle\int_{\partial D} F \cdot d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\displaystyle = \int_{\partial D} P\,dx + Q\,dy$

Example 2. *Find the line integral*

$$\int_C y^2 dx + (3xy) dy$$

over a curve C that moves, counterclockwise, around the boundary of the upper-half unit disk D.



[Method 1] By Green's theorem,

$P = y^2$

$Q = 3xy$.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y - 2y = y$.

D:

$-1 \leq x \leq 1$.

$0 \leq y \leq \sqrt{1-x^2}$.

$$\int_C y^2\, dx + (3xy)\, dy \overset{\text{Green's}}{=} \iint_D y \; dA$$

$$= \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} y \; dy \; dx$$

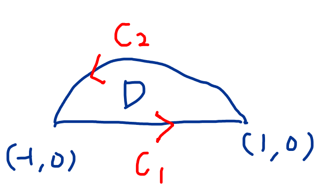
$$= \int_{-1}^{1} \frac{1}{2} y^2 \Big|_0^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1} \frac{1}{2}(1-x^2)\, dx$$

$$= \frac{2}{3}. \quad \#$$

[Method 2] By using the definition of line integrals directly.

$F = (\underline{y}^2, 3xy)$.



$$\int_C F \cdot d\vec{s} = \int_{C_1} F \cdot d\vec{s} + \int_{C_2} F \cdot d\vec{s}.$$

starting

$(-1,0) + t\left((1,0) - (-1,0)\right)$

① $C_1 : \quad c_1(t) = (-1+2t,\ 0), \quad 0 \le t \le 1.$

$$\int_{C_1} F \cdot ds = \int_0^1 F(c_1(t)) \cdot c_1'(t)\, dt$$

$$= \int_0^1 (0,0) \cdot (2,0)\, dt = 0.$$

② $C_2 : \quad C_2(t) = (\cos(t),\ \sin(t)), \quad 0 \le t \le \pi.$

$$\int_{C_2} F \cdot ds = \int_0^{\pi} F(c_2(t)) \cdot c_2'(t)\, dt$$

$$= \int_0^{\pi} (\sin^2 t,\ 3\sin t \cos t) \cdot (-\sin t,\ \cos t)\, dt.$$

$$= \int_0^{\pi} \left(-\sin^3 t + 3 \sin t \cos^2 t\right) dt$$

$$= \int_0^{\pi} \left(-\sin t(1-\cos^2 t) + 3 \sin t \cos^2 t\right) dt$$

$$= \int_0^{\pi} -\sin t + 4 \sin t \cos^2 t\ dt.$$

$$= \cos t + \left(-\tfrac{4}{3}\cos^3 t\right)\Big|_0^{\pi} = 2/3.$$

$$\int_C F \cdot ds = 2/3.$$ #

§Compute the area of a region by applying Green's Theorem

Recall that **the area of the region** D is

$$\text{area}(D) = \int\int_D 1 \, dA$$

Find a vector field $F = (P, Q)$ so that

$$\underline{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.}$$

We can choose

$$P = -y/2, \quad Q = x/2.$$

Then by Green's theorem,

$$\text{area}(D) = \iint_D 1 \, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

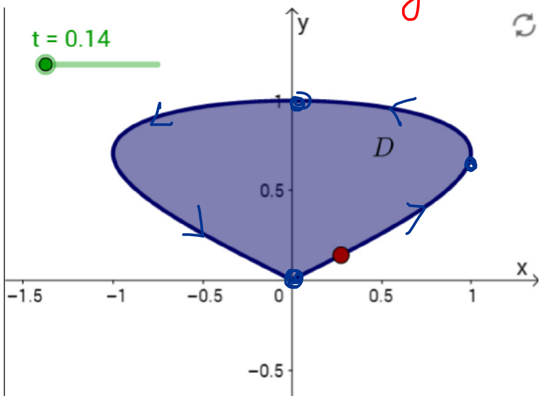
Green's theorem

$$= \int_{\partial D} P \, dx + Q \, dy$$

$$= \int_{\partial D} \frac{-y}{2} \, dx + \frac{x}{2} \, dy$$

<u>Fact</u>: $\boxed{\text{area}(D) = \frac{1}{2} \int_{\partial D} -y \, dx + x \, dy.}$

Example 3. *Find the area of the region D enclosed by the curve C parametrized by* $c(t) = \underbrace{\sin(2t)}_{x}\boldsymbol{i} + \underbrace{\sin(t)}_{y}\boldsymbol{j}$, $0 \le t \le \pi$.



In order to Green's theorem, we need to make sure $C(t)$ is positively oriented.

Check: $C(0) = (0, 0)$

$C(\pi/4) = \left(1, \frac{\sqrt{2}}{2}\right)$.

$C(\pi/2) = (0, 1)$.

By fact above

$$\text{area}(D) = \frac{1}{2}\int_C -y\,dx + x\,dy$$

$$= \frac{1}{2}\int_0^{\pi}\left[-\sin t\left(\frac{d}{dt}\sin(2t)\right) + \sin(2t)\left(\frac{d}{dt}\sin t\right)\right]dt$$

$$= \frac{1}{2}\int_0^{\pi}\left(-\sin t\left(2\underbrace{\cos(2t)}_{\cos^2 t - \sin^2 t}\right) + \underbrace{\sin(2t)}_{2\sin t\cos t}\cos t\right)dt$$

$$= \int_0^{\pi}\underbrace{\sin^3 t}_{\sin t\,(1-\cos^2 t)}\,dt$$

$$= 4/3. \quad \#$$