Quick Review from previous lecture

Fact. *(Green's Theorem)*
Let D be a simple region. Suppose $P : D \to \mathbb{R}$, $Q : D \to \mathbb{R}$ are of class C^1. Then

$$\int_{\partial D} P dx + Q dy = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

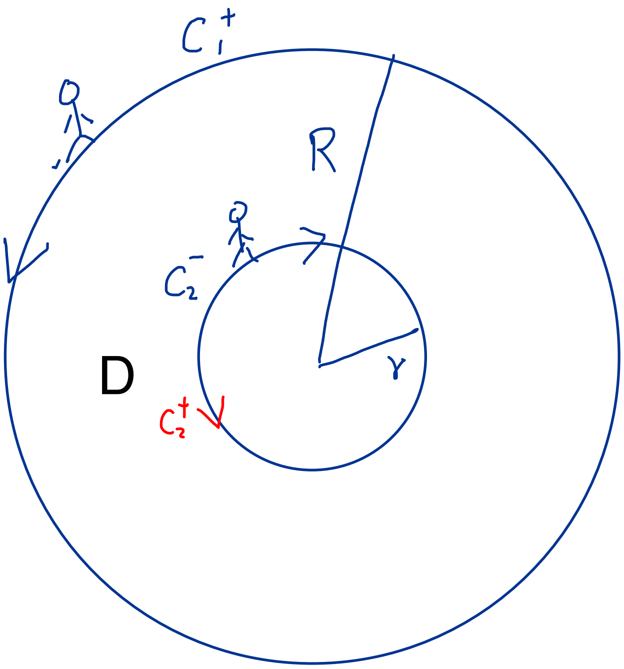
Fact. *Let D be a simple region. Then*

$$area(D) = \frac{1}{2} \left(\int_{\partial D} -y dx + x dy \right).$$

§Green's Theorem with multiple boundary components

Green's Theorem can also be applied to non-simple region or has holes in the region:

For example, we consider the annular $D: \ r^2 \leq x^2 + y^2 \leq R^2$, that has a hole in it.



make sure the region D is on your left!

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

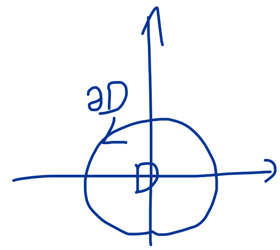
$$= \int_{C_1^+} P\,dx + Q\,dy + \int_{C_2^-} P\,dx + Q\,dy.$$

$$= \int_{C_1^+} P\,dx + Q\,dy - \int_{C_2^+} P\,dx + Q\,dy.$$

Fact: $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1^+} P\,dx + Q\,dy - \int_{C_2^+} P\,dx + Q\,dy$

$$= \int_{C_1^+} F \cdot d\vec{s} - \int_{C_2^+} F \cdot d\vec{s}$$

Example 4. *Let* $F(x,y) = (\overset{P}{y}, \overset{Q}{0})$*. Find the circulation of a vector field* F *around a unit disk with counterclockwise oriented boundary. Calculate it by using Green's theorem.*

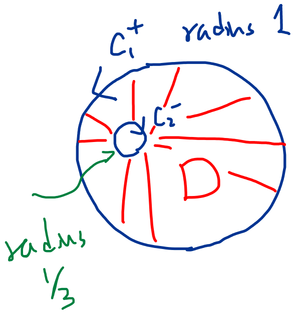


$$\text{circulation} = \int_{\partial D} F \cdot d\vec{s} \overset{\text{Green's Theorem}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (0 - 1)\, dA$$

$$= -\iint_D 1\, dA \quad \left(= -\text{area}(D)\right).$$

$$= -\pi$$



Consider $F = (y, 0)$. $\quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$$\int_{\partial D} F \cdot d\vec{s} \overset{\text{Green's}}{=} \iint_D (-1)\, dA$$

$$= -\text{area}(D).$$

$$= -\pi\left(1^2 - \left(\tfrac{1}{3}\right)^2\right) = -\left(\tfrac{8}{9}\right)\pi$$

8.3 Conservative vector fields

Conservative vector fields implies that the work done by this conservative force field does NOT depend on the path taken from point a to point b. We call it the path-independent or conservative vector field.

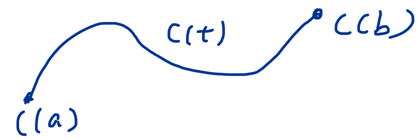
§The gradient theorem for line integrals

if $F = \nabla f$

We call a vector field F is a **gradient vector field** if $F = \nabla f$, for some scalar valued function f, that is, the vector field F can be expressed as

$\int_c F \cdot ds = f(c(b)) - f(c(a)).$

$$F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$



Fact (Gradient theorem for line integrals)**.** *If $c(t)$, $a \le t \le b$ is a path, and f is a scalar-valued function, then*

$$\int_c F \cdot d\boldsymbol{s} = \int_c \nabla f \cdot d\boldsymbol{s} = f(\overbrace{c(b)}^{\text{end}}) - f(\overbrace{c(a)}^{\text{start point}}). \tag{1}$$

We also call this kind of vector field F is **conservative** and call f is the **potential function of** F. $(F = \nabla f)$

Remark:

- Recall in Cal. 1, we have known "Fundamental theorem of Calculus (FTC)":

$$\int_a^b G'(x)dx \; \left(= G\Big|_a^b\right) = G(b) - G(a).$$

  Thus, (1) can be viewed as a generalization of FTC for more variables.

- If we can recognize the vector field F in the line integral is actually a gradient, then evaluation of the integral becomes much "easier" by using the gradient theorem for line integrals (1).

§ How to determine if a vector field F is conservative?

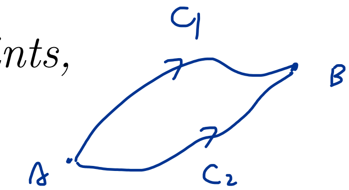
$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}$ are continuous.

Fact. *(Conservative vector fields) page 453 in textbook*
Let F be a C^1 vector field defined in $\mathbb{R}^3$, except for possibly a finite number of points. The following conditions on F are equivalent:

(1) For any oriented simple closed curve C, $\int_C F \cdot d\boldsymbol{s} = 0$.

(2) If two oriented simple curves C_1, C_2 have same endpoints,

$$\int_{C_1} F \cdot d\boldsymbol{s} = \int_{C_2} F \cdot d\boldsymbol{s}.$$



(3) There exists a scalar function f such that $F = \nabla f$.

(4) $\nabla \times F = 0$.

Remark:

- If a vector field F satisfies one of $(1) - (4)$, then F is called "**a conservative field**".
- $F = \nabla f$, then f is called "**a potential function of** F".

§The planar case ($\mathbb{R}^2$)

Fact. *(Conservative vector fields)*
Let F be a C^1 vector field defined in $\mathbb{R}^2$. The following conditions on F are equivalent:

(1) For any oriented simple closed curve C, $\int_C F \cdot d\boldsymbol{s} = 0$.

(2) If two oriented simple curves C_1, C_2 have same endpoints,

$$\int_{C_1} F \cdot d\boldsymbol{s} = \int_{C_2} F \cdot d\boldsymbol{s}.$$

(3) There exists a scalar function f such that $F = \nabla f$.

(4) $\nabla \times F = 0$.

Remark:

- In $\mathbb{R}^2$, if $F = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, then $\nabla \times F = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$
- In $\mathbb{R}^2$, the vector field F needs to be C^1 everywhere, no exception points.

Example 1. *Determine whether the following vector field is a conservative vector field.*

1. $F = e^{xy}\boldsymbol{i} + e^{x+y}\boldsymbol{j}$
2. $F = (2x\cos(y))\boldsymbol{i} - (x^2\sin(y))\boldsymbol{j}$.

① $\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & e^{x+y} & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial}{\partial x}(e^{x+y}) - \frac{\partial}{\partial y}(e^{xy}) \right\rangle$

$= \langle 0, 0, e^{x+y} - xe^{xy} \rangle$

$\neq \langle 0, 0, 0 \rangle$.

So F is NOT conservative vector field.

② $\text{curl } F = \left(\frac{\partial}{\partial x}(-x^2\sin y) - \frac{\partial}{\partial y}(2x\cos(y)) \right) k \quad (k = (0,0,1))$

$= (-2x\sin y + 2x\sin y)\, k = 0k = \langle 0, 0, 0 \rangle$

So, F is conservative.

Example 2. *Does the integral*

$$\int_C \underline{(x^2 - ze^y)}dx + \underline{(y^3 - xze^y)}dy + \underline{(z^4 - xe^y)}dz = \int F \cdot ds.$$

depend on the specific path C that we take?

vector field $F = \langle x^2 - ze^y, y^3 - xze^y, z^4 - xe^y \rangle$.

$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - ze^y & y^3 - xze^y & z^4 - xe^y \end{vmatrix}$

$= \langle -xe^y + xe^y, -e^y + e^y, -ze^y + ze^y \rangle$.

$= \langle 0, 0, 0 \rangle$.

F is conservative, so the line integral is path-independent!

Example 3. *Let* $F(x,y) = (\overbrace{2xye^{x^2}}^{P}, \overbrace{e^{x^2}}^{Q})$. *Let* $c(t) = (\cos(t), t^2+1)$, $0 \le t \le 1$. *Is* F *a conservative vector field? Find* $\int_c F \cdot d\boldsymbol{s}$.

1. Show F is conservative.

$$\text{curl } F = \left(\frac{\partial}{\partial x} e^{x^2} - \frac{\partial}{\partial y}(2xy\, e^{x^2}) \right) k = \left(2xe^{x^2} - 2xe^{x^2}\right) k$$

$$= 0k.$$

$$= (0,0,0)$$

2. We can find a function f such that $F = \nabla f$.

To find f.

$f_x = 2xy\,e^{x^2} \Longrightarrow f = ye^{x} + \underline{g(y)}$ independent of x

$f_y = e^{x^2} \Longrightarrow f = ye^{x^2} + \underline{h(x)}$ independent of y.

Chosing $g(y) = h(x) = 0$. Thus,

$$f(x,y) = ye^{x^2}.$$

By gradient theorem,

$$\int F \cdot ds = \int \nabla f \cdot ds = f(c(1)) - f(c(0))$$

$$= f(\cos(1), 2) - f(1,1)$$

$$= 2e^{\cos^2(1)} - e^1. \#$$