Quick Review from previous lecture

- Gradient Theorem:

$$\int_c F \cdot d\mathbf{s} = \int_c \nabla f \cdot d\mathbf{s} = f(c(b)) - f(c(a)). \tag{1}$$

- **Fact.** *(Conservative vector fields)*
  Let F be a C^1 vector field defined in $\mathbb{R}^3$, except for possibly a finite number of points. The following conditions on F are equivalent:

  (1) For any oriented simple closed curve C, $\int_C F \cdot d\boldsymbol{s} = 0$.

  (2) If two oriented simple curves C_1, C_2 have same endpoints,

$$\int_{C_1} F \cdot d\boldsymbol{s} = \int_{C_2} F \cdot d\boldsymbol{s}.$$

  (3) There exists a scalar function f such that $F = \nabla f$.

  (4) $\nabla \times F = 0$.

Today's topics

- Polar, cylindrical, spherical coordinates.

Example 4. *Let $F(x,y,z) = (2xyz + \sin(x),\ x^2z,\ x^2y)$. Let $c(t) = (1,\ t,\ t^2)$, $0 \le t \le 1$. Is F a conservative vector field? If so, find a potential for F. In addition, evaluate $\int_c F \cdot d\mathbf{s}$.*

1. (curl F)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz+\sin(x) & x^2z & x^2y \end{vmatrix} = \langle x^2 - x^2,\ 2xy - 2xy,\ 2xz - 2xz \rangle$$

$$= \langle 0,\ 0,\ 0 \rangle.$$

So, F is conservative.

2. We can find a potential f such that $F = \nabla f$.

$$\frac{\partial f}{\partial x} = f_x = 2xyz + \sin x \Rightarrow f = x^2yz - \cos(x) + \underline{c_1(y,z)}$$ independent of x.

$$\frac{\partial f}{\partial y} = f_y = x^2 z \Rightarrow f = x^2yz + \underline{c_2(x,z)}$$ $\rightarrow$ independent of y.

$$\frac{\partial f}{\partial z} = f_z = x^2 y \Rightarrow f = x^2yz + \underline{c_3(x,y)}$$ $\rightarrow$ independent of z.

Taking $c_1(y,z) = 0$, $c_2(x,z) = -\cos x$
$c_3(x,y) = -\cos x$,

Then $f = x^2yz - \cos(x)$

3. By Gradient Theorem,

$$\int F \cdot ds = \int \nabla f \cdot ds = f(c(1)) - f(c(0)).$$

$$= f(1,1,1) - f(1,0,0)$$

$$= 1 - \cos(1) + \cos 1 = 1.$$ #

Fact. *If F is a C^1 vector field in $\mathbb{R}^3$, then*

$$\nabla \cdot F = 0 \Leftrightarrow F = \nabla \times G \quad \textit{for some } C^1 \textit{ vector field } G.$$

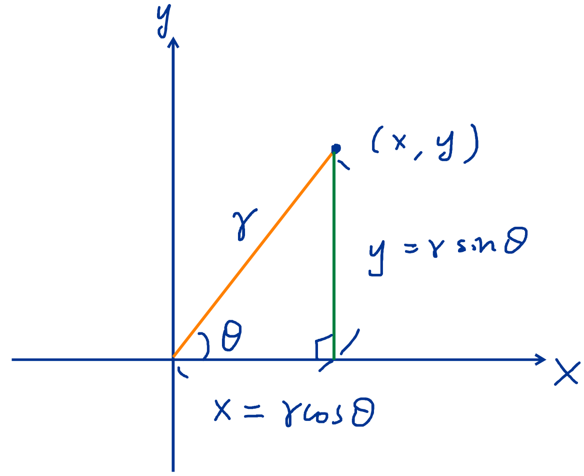
$\nabla \times F = 0 \iff F = \nabla f$, f scalar function.

1.4 Cylindrical and spherical coordinates

A point in the plane R^2 can be represented by Cartesian coordinate (x, y) and polar coordinate (r, θ). These coordinates are related by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$.



On the other hand,

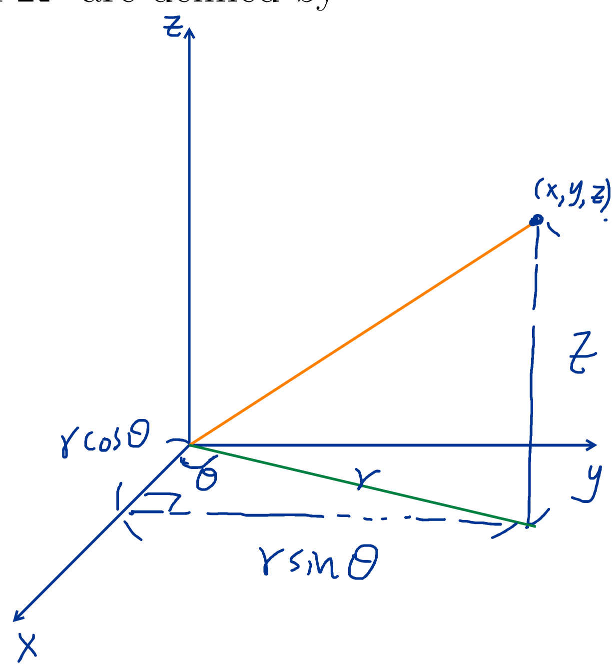
$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x},\ \ x \neq 0$$

Definition: Cylindrical coordinates

Cylindrical coordinates (r, θ, z) of a point (x, y, z) in $\mathbb{R}^3$ are defined by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

where $r \geq 0$ and $0 \leq \theta < 2\pi$.



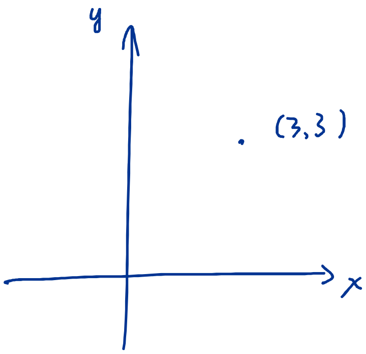
On the other hand,

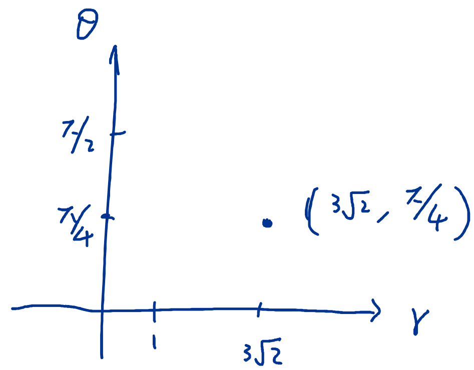
$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x},\ \ x \neq 0$$
$$z = z$$

Example 1. *Find and plot the polar coordinate of Cartesian coordinate point* $(3, 3)$.

$$r = \sqrt{3^2+3^2} = 3\sqrt{2}.$$

$$\tan\theta = \frac{y}{x} = \frac{3}{3} = 1 \Rightarrow \theta = \pi/4.$$





Example 2. *Find and plot the cylindrical coordinate of Cartesian coordinate point* $(-4, 4\sqrt{3}, 5)$.

$$r = \sqrt{(-4)^2+(4\sqrt{3})^2} = \sqrt{16+48} = \sqrt{64} = 8.$$

$$\tan\theta = \frac{y}{x} = -\sqrt{3}. \Rightarrow \theta = \frac{2\pi}{3}.$$

$$\left(8, \frac{2\pi}{3}, 5\right).$$

Definition: Spherical coordinates

For a point (x, y, z) in $\mathbb{R}^3$, take

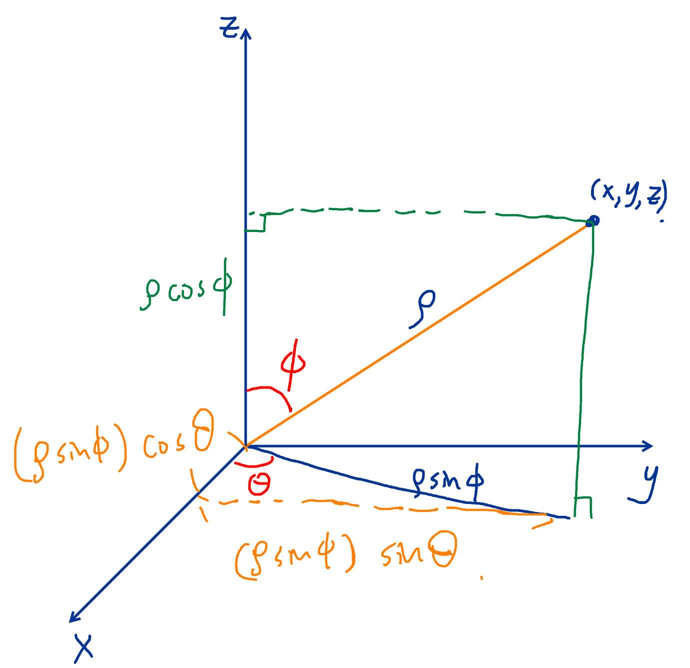
$$\boxed{\rho = \sqrt{x^2 + y^2 + z^2}}$$
$$0 \le \theta < 2\pi$$
$$0 \le \phi \le \pi$$

Spherical coordinates (ρ, θ, ϕ) of a point (x, y, z) in $\mathbb{R}^3$ are defined by

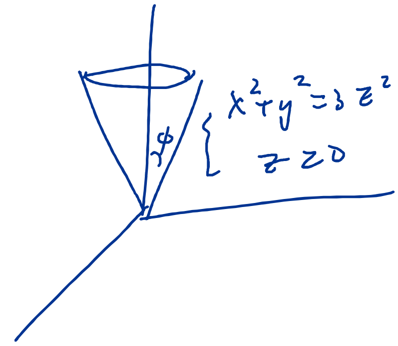
$$\boxed{\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}}$$

We also have the following relations:

$$\boxed{\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2} \\ \tan\theta &= \frac{y}{x}, \;\; x \ne 0 \\ \cos\phi &= \frac{z}{\rho}, \;\; \rho \ne 0 \end{aligned}}$$



Example 3. *1. What is the surface $\phi = \pi/3$?*



$$\begin{cases} x = \rho\sin\phi\cos\theta = \rho\frac{\sqrt{3}}{2}\cos\theta \\ y = \rho\sin\phi\sin\theta = \rho\frac{\sqrt{3}}{2}\sin\theta \\ z = \rho\cos\phi = \rho\frac{1}{2} \end{cases}$$

$$\begin{cases} x^2+y^2 = \rho^2\frac{3}{4} \\ z^2 = \frac{\rho^2}{4} \end{cases} \qquad \begin{cases} x^2+y^2 = 3z^2. \\ z \ge 0. \end{cases}$$

2. What is the surface $\theta = \pi/3$?

$$\begin{cases} x = \rho\sin\phi\cos\theta & \Rightarrow x = \rho\sin\phi\left(\frac{1}{2}\right) \\ y = \rho\sin\phi\sin\theta & \Rightarrow y = \rho\sin\phi\frac{\sqrt{3}}{2} \\ z = \rho\cos\phi & \Rightarrow z = \rho\cos\phi. \end{cases} \Rightarrow \begin{array}{l} \sqrt{3}x = y \\ x \ge 0 \\ y \ge 0. \end{array}$$

3. What is the surface $\rho = 3$?

$$x^2+y^2+z^2$$

$$= \underbrace{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}_{} + \rho^2\cos^2\phi$$

$$= \rho^2\sin^2\phi + \rho^2\cos^2\phi$$

$$= \rho^2 = 9.$$

sphere with radius 3.