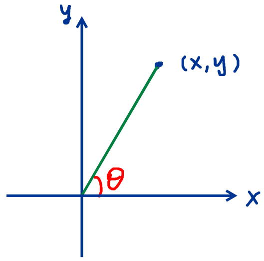
Quick Review from last week

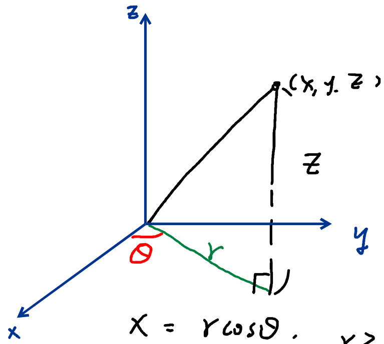
§ Polar coordinate:



$x = r\cos\theta$, $0 \le r$

$y = r\sin\theta$, $0 \le \theta < 2\pi$

§ Cylindrical coordinate:

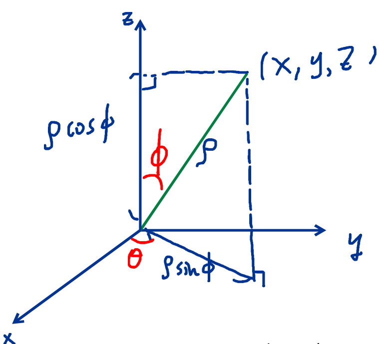


$x = r\cos\theta$, $r \geq 0$

$y = r\sin\theta$, $0 \le \theta < 2\pi$

$z = z$,

§ Spherical coordinate:



$x = (\rho\sin\phi)\cos\theta$, $\rho \geq 0$

$y = (\rho\sin\phi)\sin\theta$, $0 \le \theta < 2\pi$

$z = \rho\cos\phi$, $0 \le \phi \le \pi$

- Suppose $\det A \neq 0$. If a map

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix},$$

then T maps **parallelograms** into **parallelograms** and **vertices** into **vertices**. On the other hand, if the image $T(D^*)$ is a parallelogram, then D^* must be a parallelogram.

Quiz 7 covers 6.1, 6.2.

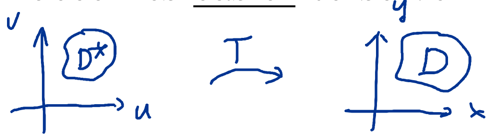
Today's topics

- Change of variables for double and triple integrals.

6.2 The change of variables theorem

One motivation to study "Change of variables", is to transform the region of integration so that the resulting integral becomes easier to solve.

§Change of variables for double integrals



Fact. *Let D^* and D be elementary regions in $\mathbb{R}^2$. Let T maps D^* onto D is given by*

$$T(u,v) = (x(u,v), y(u,v)).$$

Then

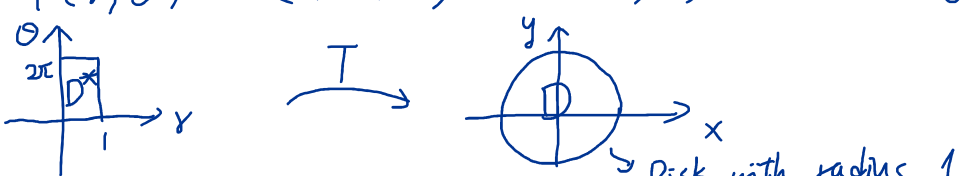
$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v),y(u,v)) \overbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}^{=\,|\det DT|} dudv.$$

Here the determinant of the derivative matrix

$$det\underbrace{\boldsymbol{D}T}_{\text{matrix of partial derivative}} = \frac{\partial(x,y)}{\partial(u,v)} = det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix},$$

*the **Jacobian** of T.*

EX : $T(r,\theta) = (r\cos\theta,\ r\cos\theta)$, $D^* = [0,1] \times [0,2\pi]$.



$$\iint_{D^*} \tilde{f}(r,\theta)\underbrace{\boxed{\text{Jacobian}}}_{=\,r} dr d\theta = \iint_D f(x,y)\,dx\,dy$$

Example 1. *Consider the map T which transforms polar coordinates into Cartesian coordinates. Then $T(r,\theta) = (r\cos\theta,\ r\sin\theta)$, that is,*

$$x = r\cos\theta, \quad y = r\sin\theta.$$

What is Jacobian of T?

$$\text{Jacobian of } T = det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

$$= det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r. \ \#$$

Fact. *(Change of variables - Polar coordinates)*

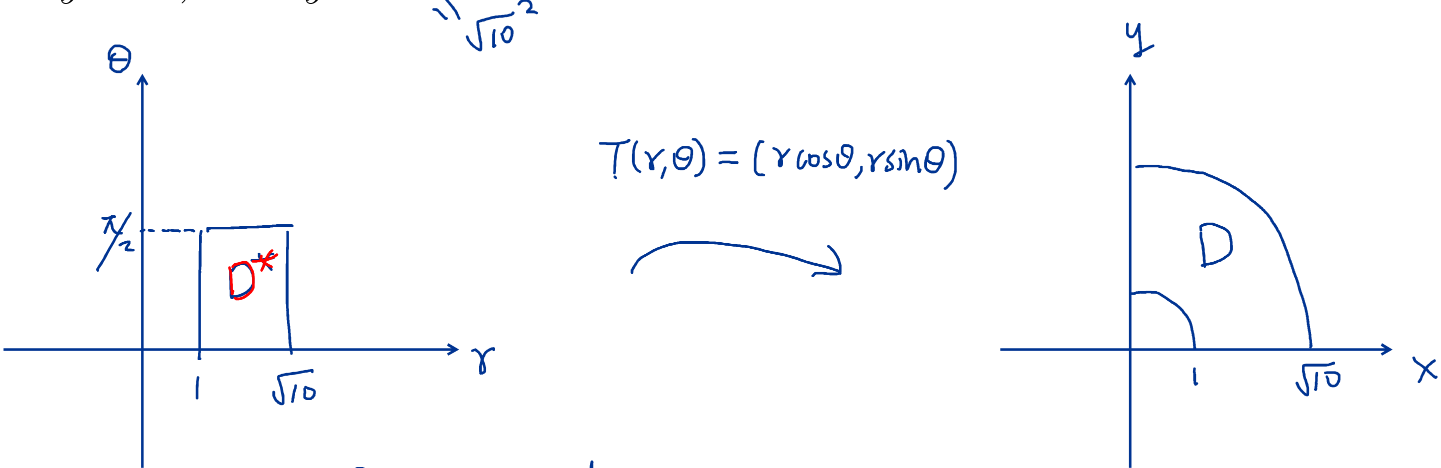
$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(r\cos\theta,\ r\sin\theta)\ r\ drd\theta.$$

Example 2. *Evaluate*

$$\int\int_D \sin(x^2+y^2) + e^{2x^2+2y^2} dxdy,$$

where D is the region in the first quadrant of the xy-plane lying between $x^2+y^2=1$, $x^2+y^2=10$.

$= \sqrt{10}^2$



what is D^* ? rectangle

$1 \le r \le \sqrt{10}$

$0 \le \theta \le \pi/2$

$\iint_D f(x,y)\,dx\,dy$

By change of variables,

$$\iint_D \sin(x^2+y^2) + e^{2x^2+2y^2}\,dx\,dy = \iint_{D^*} \left(\sin(r^2) + e^{2r^2}\right) \overbrace{r}^{\text{Jacobian.}}\, dr\,d\theta$$

or use $u=r^2$

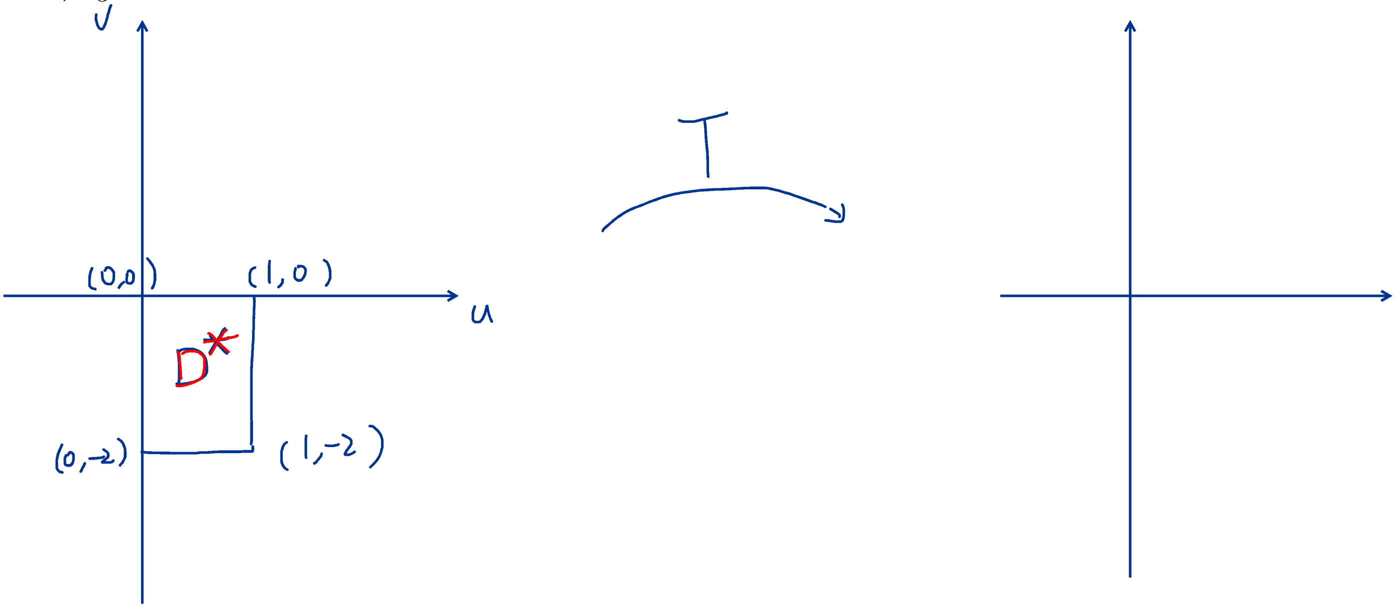
$$= \int_0^{\pi/2}\int_1^{\sqrt{10}} \left(r\sin(r^2) + re^{2r^2}\right) dr\,d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}\cos(r^2) + \frac{1}{4}e^{2r^2}\Big|_1^{\sqrt{10}}\, d\theta$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}\cos(10) + \frac{1}{2}\cos 1 + \frac{1}{4}e^{20} - \frac{1}{4}e^2\right)$$

#

Example 3. *Let* $T(u,v) = (\overbrace{u-v}^{x},\ \overbrace{2u-v}^{y})$. *Let* $D^* = [0,1]\times[-2,0]$. *Let* $D = T(D^*)$. *Evaluate* $\int\int_D xydxdy$ *by making the change of variables* $x = u-v,\ y = 2u-v$.



$$\iint_D xy\,dx\,dy$$

By change of variables,

$$\iint_D xy\,dx\,dy = \iint_{D^*} (u-v)(2u-v)\overbrace{1}^{\text{Jacobian}}\,du\,dv$$

$$\text{Jacobian} = \det\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = -1+2 = 1.$$

$$= \int_{-2}^{0}\int_{0}^{1} (u-v)(2u-v)\,du\,dv$$

$$= \int_{-2}^{0}\int_{0}^{1} (2u^2-3uv+v^2)\,du\,dv = 7. \quad \#$$

Q: what is D? parallelogram!

$$T(u,v) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u-v \\ 2u-v \end{bmatrix}$$

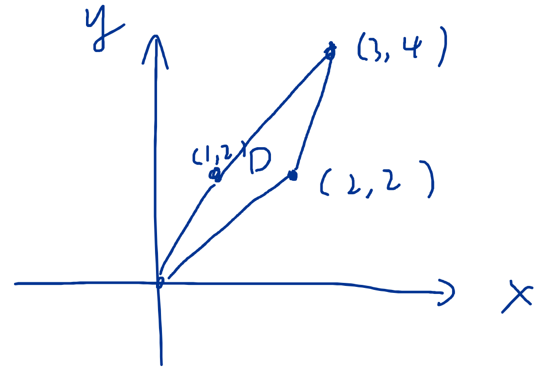
$\det\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = 1 > 0$. T maps parallelogram into parallelogram. vertices into vertices

$T(1,0) = (1,2)$

$T(1,-2) = (3,4)$

$T(0,-2) = (2,2)$

$T(0,0) = (0,0).$



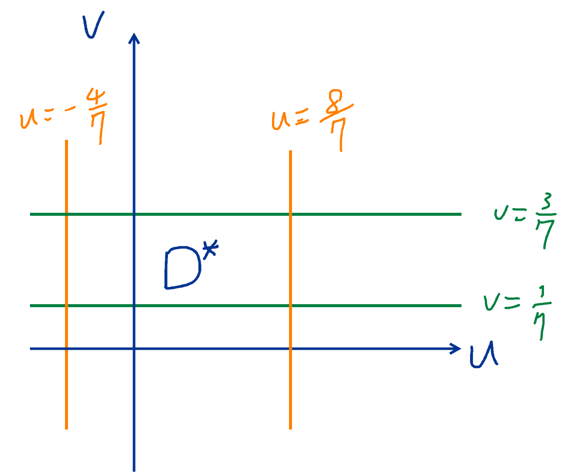
Example 4. *Evaluate the integral* $\int\int_D (3x-2y)\,dA$ *where the parallelogram* D *is bounded by the lines*

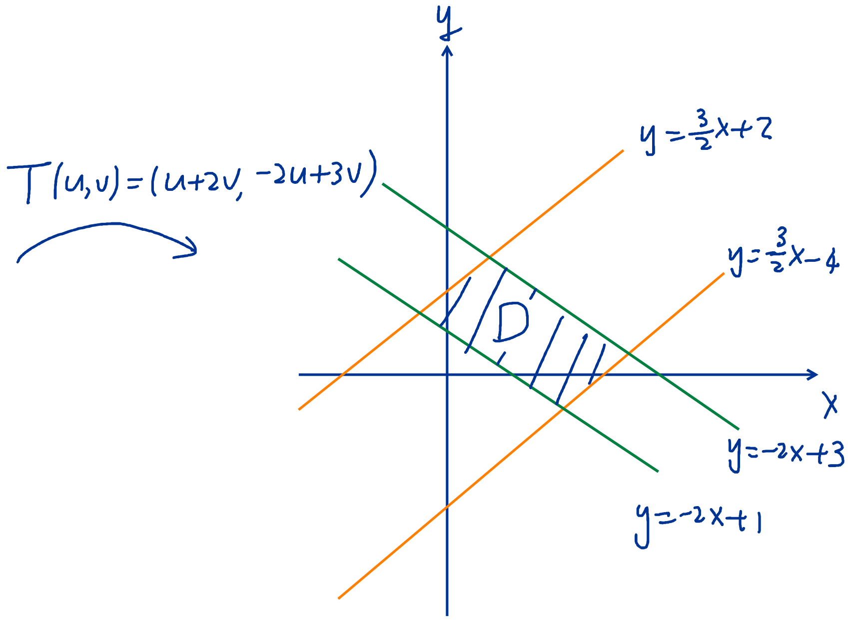
$$y = \frac{3}{2}x - 4,\ \ y = \frac{3}{2}x + 2,\ \ y = -2x + 1,\ \ y = -2x + 3.$$

Use the linear change of variables

$$(x, y) = T(u, v) = (\overbrace{u + 2v}^{x},\ \overbrace{-2u + 3v}^{y})$$

to solve this problem. [See EX 6 in MTE 03 23 2018]





D^* : $-\frac{4}{7} \le u \le \frac{8}{7}$

$\frac{1}{7} \le v \le \frac{3}{7}$

By change of variables,

$$\iint_D (3x-2y)\,dA = \iint_{D^*} \left[3(u+\cancel{2v}) - 2(-2u+\cancel{3v})\right] \overbrace{7}^{\text{Jacobian}} \,du\,dv$$

$$\text{Jacobian} = \det\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} = 3 + 4 = 7$$

$$= \int_{1/7}^{3/7}\int_{-4/7}^{8/7} 7u\,(7)\,du\,dv$$

$$= \int_{1/7}^{3/7}\int_{-4/7}^{8/7} 49u\,du\,dv = 48/7.$$

§Change of variables for Triple integrals

Fact. *Let W^* and W be elementary regions in $\mathbb{R}^3$. Suppose that*

$$T : W^* \to W$$

is given by

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)).$$

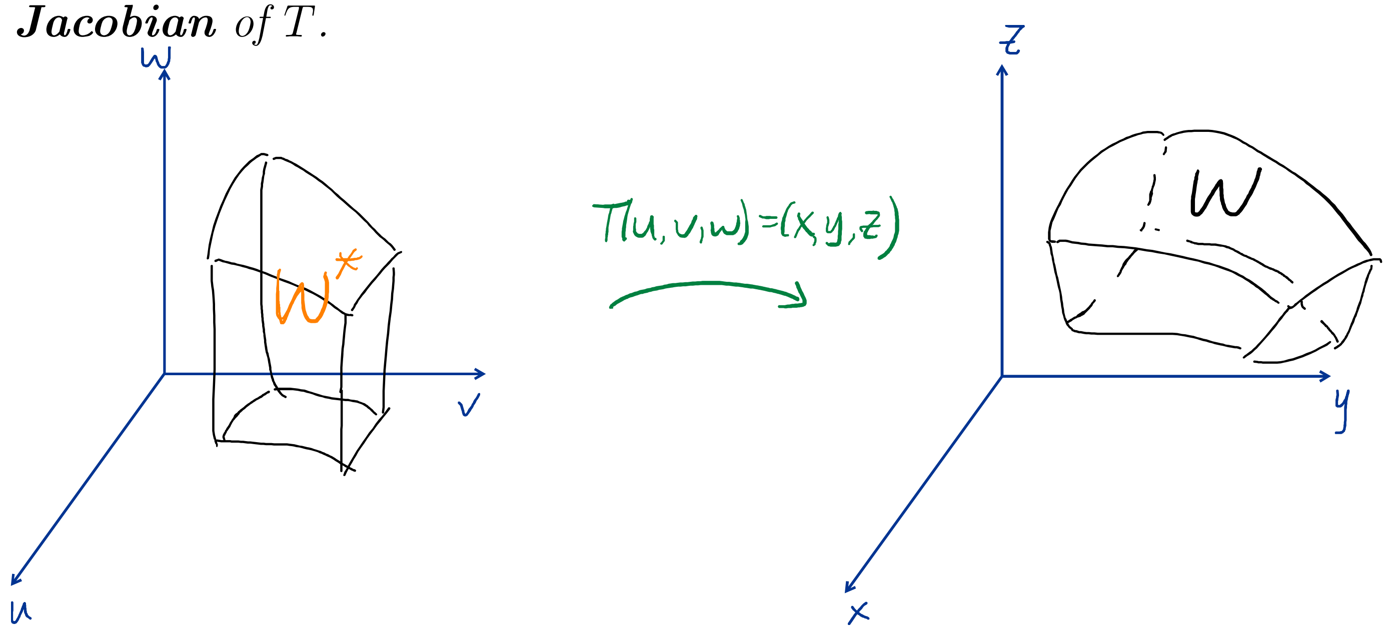
Then

$$\iiint_W f(x,y,z)\,dxdydz$$
$$= \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \overbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}^{|\det DT|} dudvdw.$$

Here the determinant of the derivative matrix

$$det\boldsymbol{D}T(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix},$$

*the **Jacobian** of T.*



$$\iiint_{W^*} \tilde{f}(u,v,w)\,|\text{Jacobian}|\,dudvdw \underset{\substack{\text{change}\\ \text{of}\\ \text{variables}}}{=} \iiint_W f(x,y,z)\,dxdydz$$

Fact. *(Change of variables - Cylindrical coordinates)*

$T(r,\theta,z)$

$=(r\cos\theta, r\sin\theta, z)$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix}$$

where $r \geq 0$ *and* $0 \leq \theta < 2\pi$. *Then*

$$\int\int\int_W f(x,y,z)dxdydz = \int\int\int_{W^*} f(r\cos\theta,\ r\sin\theta, z) \underbrace{r}_{Jacobian} drd\theta dz.$$

$$= \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= r.$$

$T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

Fact. *(Change of variables - Spherical coordinates)*

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \Rightarrow \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} =$$

where $\rho \geq 0$ *and* $0 \leq \theta < 2\pi$ *and* $0 \leq \rho \leq \pi$. *Then*

$$= \det\begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

$$= -\rho^2\sin\phi.$$

$$\int\int\int_W f(x,y,z)dxdydz$$

$$= \int\int\int_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \underbrace{\rho^2\sin\phi}_{Jacobian} d\rho d\theta d\phi.$$

$$\parallel$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \left|-\rho^2\sin\phi\right|$$

$$= \rho^2\sin\phi$$