# Math 2374 Spring 2018 - Week 11

## Quick Review from previous lecture

- $$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v),y(u,v))\left|\det \mathbf{D}T(u,v)\right| dudv.$$

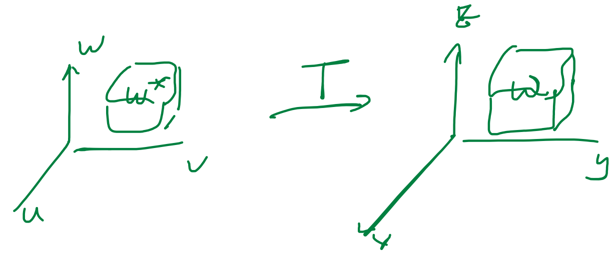
  Here the determinant of the derivative matrix

  $$\det \mathbf{D}T(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix},$$

  the **Jacobian** of $T$. Thus, we also have the following expression

  $$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

  Polar coord. : $\iint \quad r\, dr d\theta.$
  $x = r\cos\theta$, $y = r\sin\theta$.




- $$\int\int\int_W f(x,y,z)dxdydz$$
  $$= \int\int\int_{W^*} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw.$$

  Here the determinant of the derivative matrix

  $$\det \mathbf{D}T(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix},$$

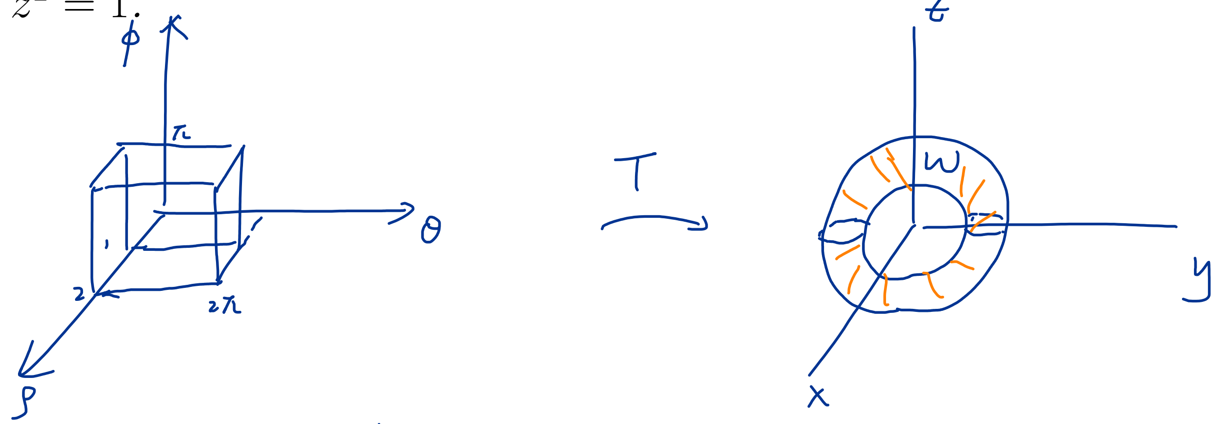
  the **Jacobian** of $T$.

  - Cylindrical coord. $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.
    $\iiint_{W^*} \cdots \quad r\, dr d\theta dz.$
  - spherical coord. $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.
    $\iiint_{W^*} \cdots \quad \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$

**Example 5.** *Evaluate*

$$\iiint_W \sqrt{x^2+y^2+z^2}\; e^{-(x^2+y^2+z^2)}\,dV,$$

*where W is the solid region bounded by two sphere* $x^2+y^2+z^2 = \overset{2^3}{\overset{\shortparallel}{4}}$ *and* $x^2+y^2+z^2=1$.




Using spherical coordinates,

$$T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi).$$

$$1 \le \rho \le 2,\quad 0 \le \theta < 2\pi,\quad 0 \le \phi \le \pi.$$

By changing of variables,

$$\iiint_W \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\,dV$$

$$= \int_0^{\pi}\int_0^{2\pi}\int_1^2 \rho\, e^{-\rho^2}\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

$$= \int_0^{\pi}\int_0^{2\pi}\left(\int_1^2 \rho^3 e^{-\rho^2}\,d\rho\right)\sin\phi\, d\theta\, d\phi.$$

$u=\rho^2,\ du = 2\rho\, d\rho$

$$\int_1^2 \rho^3 e^{-\rho^2}\,d\rho = \int u\, e^{-u}\,\frac{du}{2}.$$

Integration by parts.

$$= {}^{8}\frac{1}{2}\left(-5e^{-4}+2e^{-1}\right).$$

$$= \int_0^{\pi}\int_0^{2\pi} \left( \frac{1}{2}(-5e^{-4} + 2e^{-1}) \right) \sin\phi \; d\theta \; d\phi$$

$$= 2\pi \left( \frac{1}{2}(-5e^{-4} + 2e^{-1}) \right) \left( \int_0^{\pi} \sin\phi \; d\phi \right)$$

$$= -2\pi \left( 5e^{-4} - 2e^{-1} \right). \quad \#$$

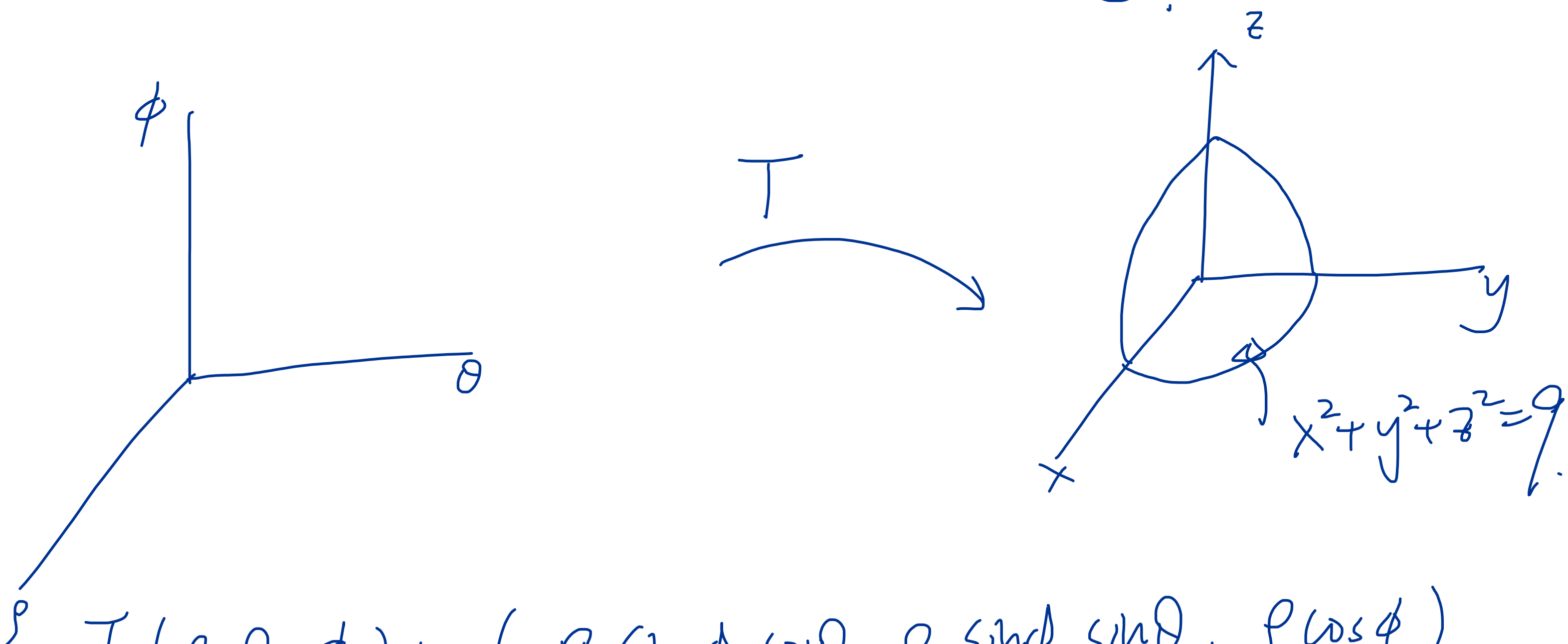
**Example 6.** *Evaluate this integral*

$$\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{9-z^2-y^2}} (10 - \sqrt{x^2+y^2+z^2})\,dx\,dy\,dz.$$

From problem, $0 \le x \le \sqrt{9-z^2-y^2}$. $[\, x = \sqrt{9-z^2-y^2}$, $x^2+z^2+y^2 = 9 \,]$

$0 \le y \le \sqrt{9-z^2}$

$0 \le z \le 3$.




$$T(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi).$$

$0 \le \rho \le 3$.

$0 \le \phi \le \pi/2$

$0 \le \theta \le \pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 (10-\rho)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left( \left(\frac{10}{3}\rho^3 - \frac{1}{4}\rho^4\right)\sin\phi \Big|_0^3 \right) d\phi\,d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left( \left(90 - \frac{1}{4} 81\right) \sin\phi \right) d\phi \, d\theta$$

$$= \int_0^{\pi/2} \left(-90 + \frac{81}{4}\right) \cos\phi \Big|_0^{\pi/2} \, d\theta$$

$$= \underline{\frac{279}{8}\pi}$$ #

# 7.3 Parametrized Surfaces

In Sec. 7.1, 7.2, we have studied integrals along curves.
Now in Sec. 7.3-7.6, we are going to learn how to do integrals over surfaces.
Let's begin by studying how to parametrize a surface.

**Parametrized surfaces** extends the idea of parametrized curves to vector-valued functions of **2** variables.

**Example:**

$c(t) = (t, t^2)$ parametrizes a parabola $y = x^2$. Notice that $c(t)$ only has **1** variable.

Similar idea, to parametrize surfaces, we need a function of **2** variables.

(1) A unit sphere centered at the origin is parametrized by the function

$$(x, y, z) = \Phi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

where $0 \le \theta < 2\pi$, and $0 \le \phi \le \pi$. ①

upper sphere $z = \sqrt{1-x^2-y^2}$. $\Phi(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

$0 \le \phi \le \pi$, $0 \le \theta < 2\pi$.

② $\Phi(x,y) = (x, y, \sqrt{1-x^2-y^2})$.

with $x^2+y^2 \le 1$.

(2) Parametrize the cone $z = \sqrt{x^2 + y^2}$.

① cylindrical coord.

$\Phi(r,\theta) = (r\cos\theta, r\sin\theta, r)$,

$0 \le r$, $0 \le \theta < 2\pi$.

② $\Phi(x,y) = (x, y, \sqrt{x^2+y^2})$, $-\infty < x, y < \infty$.

**Example 1.** *Give a parametrization of the plane*

$$2x + 7y - 15z - 123 = 0$$

*with* $ABC \neq 0$.

$$\Phi(x, y) = \left( x,\ y,\ \frac{2x+7y-123}{15} \right), \quad -\infty < x, y < \infty.$$

or $$\Phi(x, z) = \left( x,\ \frac{-2x+15z+123}{7},\ z \right), \quad -\infty < x, z < \infty.$$

**Example 2.** *Give a parametrization of the surface* $x^2+y^2 = z$ *with* $0 \leq z \leq 25$.

① Cylindrical.

$0 \le r^2 \le 25$.

$$\Phi(r, \theta) = (r\cos\theta,\ r\sin\theta,\ r^2), \quad 0 \le r \le 5, \quad 0 \le \theta \le 2\pi.$$

② $\Phi(x, y) = (x, y, x^2+y^2)$, $0 \le x^2+y^2 \le 25$

** Check math insight for various Quadric surfaces in last section in Part 19**

## §Tangent vectors to a parametrized surface

**Fact.** *Suppose*

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

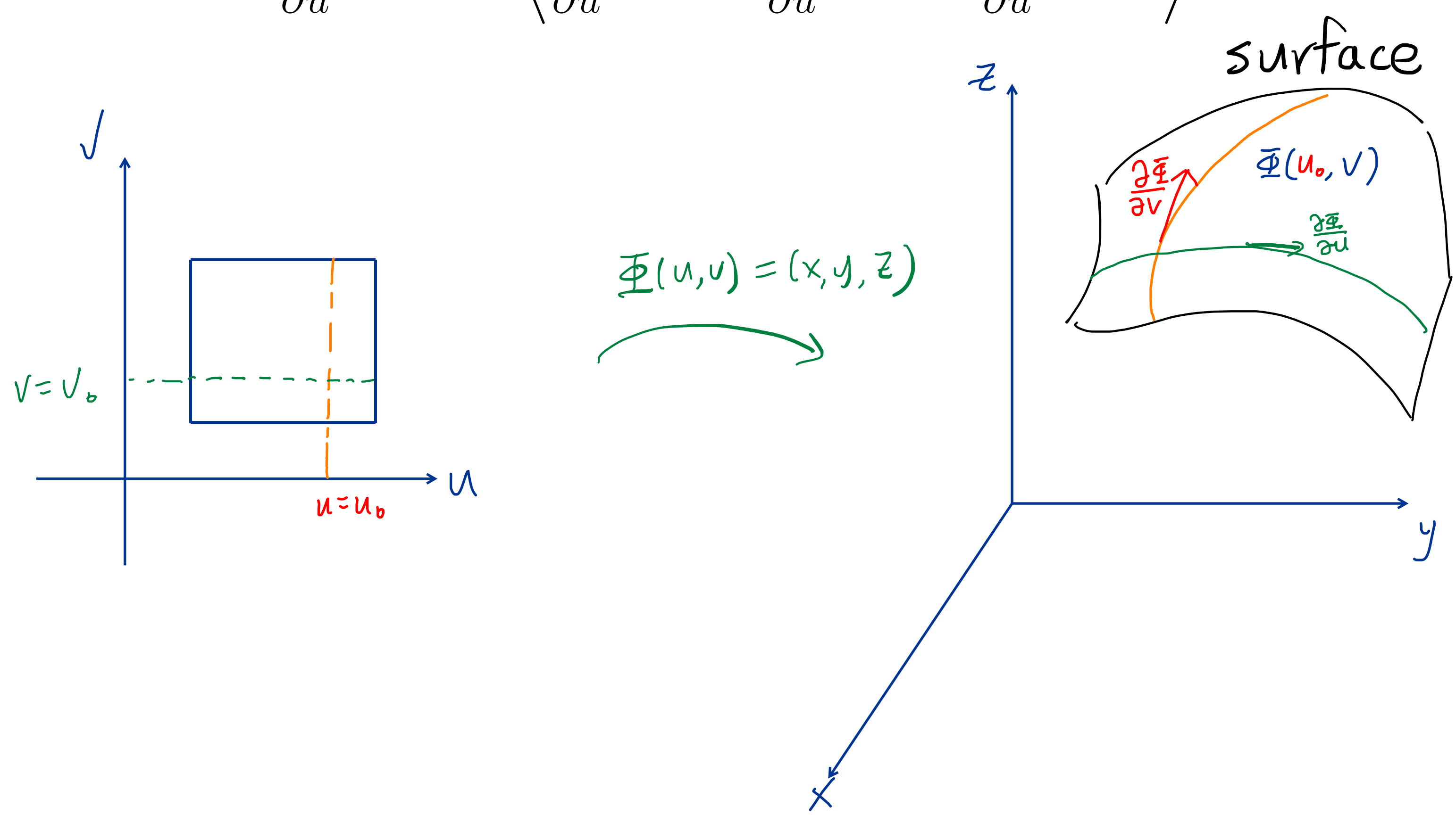
*is a parametrization of a surface $S$. Assume $\Phi$ is differentiable at $(u_0, v_0)$ from the domain.*

*(1) Fix $u = u_0$, then $\Phi(u_0, v)$ is a curve on $S$. The* ***tangent vector*** *to this curve at point $\Phi(u_0, v_0)$ is*

$$T_v := \frac{\partial \Phi}{\partial v}(u_0, v_0) = \left\langle \frac{\partial x}{\partial v}(u_0, v_0),\ \frac{\partial y}{\partial v}(u_0, v_0),\ \frac{\partial z}{\partial v}(u_0, v_0) \right\rangle$$

*(2) Fix $v = v_0$, then $\Phi(u, v_0)$ is a curve on $S$. The* ***tangent vector*** *to this curve at point $\Phi(u_0, v_0)$ is*

$$T_u := \frac{\partial \Phi}{\partial u}(u_0, v_0) = \left\langle \frac{\partial x}{\partial u}(u_0, v_0),\ \frac{\partial y}{\partial u}(u_0, v_0),\ \frac{\partial z}{\partial u}(u_0, v_0) \right\rangle$$



**Remarks:**

- $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$ are tangent to curves on surface $S$.
- Thus, a **unit normal vector** to the surface $S$ is

$$\mathbf{n} = \frac{\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}}{\|\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\|}$$

**Definition:**

We call the surface $S$ is an **oriented surface** if $S$ is a two-sided surface with one side specified as the outside or positive side; the other side as the inside or negative side.

The orientation of a surface is given by the unit normal vector $\mathbf{n}$.

**Remarks:**

- If the unit normal $\mathbf{n} = (x, y, z)$ has positive $z$ component $(z \geq 0)$, then we call $\mathbf{n}$ is the **upward-pointing normal** on $S$.
- Möbius strip is NOT oriented surface since it only has one side. See page 402.