Math 2374 Spring 2018 - Week 12

Quick Review from last week

1. $$\iint_D f(x,y)\,dxdy = \iint_{D^*} f\left(x(u,v), y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dudv$$

   Polar coordinates: $x = r\cos(\theta)$ and $y = r\sin(\theta)$, then $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$.

2. $$\iiint_W f(x,y,z)\,dxdydz = \iiint_{W^*} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,dudvdw$$

   Cylindrical coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$, then $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$.

   Spherical coordinates: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$ with $0 \le \rho$, $0 \le \phi \le \pi$, $0 \le \theta < 2\pi$, then $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2\sin(\phi)$.

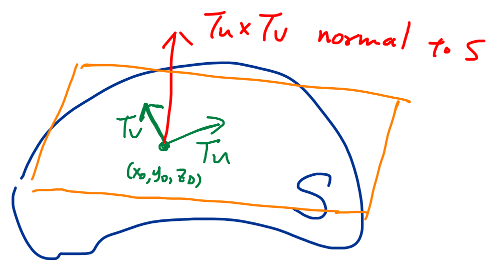
3. (Parametrization of a surface S)

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v)).$$

$T_u = \frac{\partial\Phi}{\partial u}$, $T_v = \frac{\partial\Phi}{\partial v}$

- $\frac{\partial\Phi}{\partial u}$ and $\frac{\partial\Phi}{\partial v}$ are tangent to curves on surface S.
- A **unit normal vector** to the surface S is

$$\mathbf{n} = \frac{\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}}{\|\frac{\partial\Phi}{\partial u}\times\frac{\partial\Phi}{\partial v}\|}$$

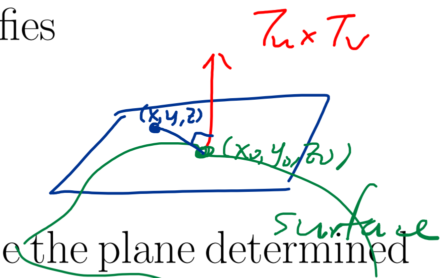


* Quiz 8 : 7.3, 7.4, 7.5.

§Tangent planes to a parametrized surface

Definition: If a parametrized surface $\Phi : D \to \mathbb{R}^3$ satisfies

$$\left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right)(u_0, v_0) \neq 0.$$



We define the **tangent plane** of the surface at $\Phi(u_0, v_0)$ to be the plane determined by $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial v}$. An equation of the tangent plane at (x_0, y_0, z_0) on the surface is given by

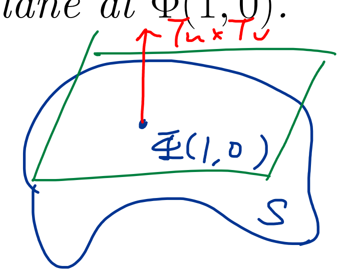
$$\boxed{(x - x_0, y - y_0, z - z_0) \cdot \left(\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}\right)(u_0, v_0) = 0.}$$

Example 3. *A given surface S is parametrized by*

$$x = u \cos v, \quad y = u \sin v, \quad z = u^2 + v^2,$$

$u=1,\ v=0.$

that is, $\Phi(u, v) = (u \cos v,\ u \sin v,\ u^2 + v^2)$. Find the tangent plane at $\Phi(1, 0)$.



$$T_u = \langle \cos v,\ \sin v,\ 2u \rangle$$

$$T_v = \langle -u \sin v,\ u \cos v,\ 2v \rangle.$$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 2v \end{vmatrix}$$

$$= \langle 2v \sin v - 2u^2 \cos v,\ -2u^2 \sin v - 2v \cos v,\ u \cos^2 v + u \sin^2 v \rangle.$$

$$(T_u \times T_v)(1, 0) = \langle -2,\ 0,\ 1 \rangle.$$

Point is $\Phi(1,0) = (1, 0, 1)$.

Tangent plane is

$$(x-1,\ y-0,\ z-1) \cdot (-2,\ 0,\ 1) = 0$$

$$\Rightarrow -2x + z + 1 = 0. \quad \#$$

7.4 Area of the surface

Let $\Phi : \ D \ \rightarrow \ \mathbb{R}^3$ be written as

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v)),$$

which is a parametrization of the surface S. Suppose $\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \neq 0$, where

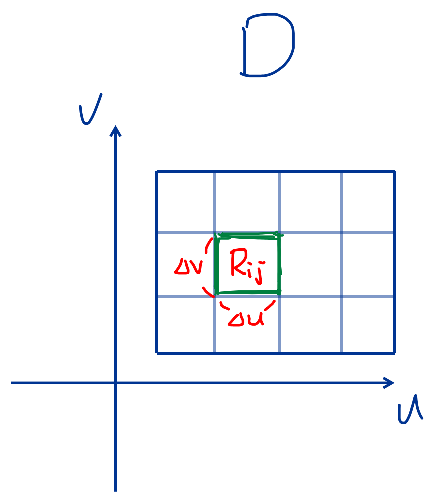
Denote
$$T_u = \frac{\partial \Phi}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

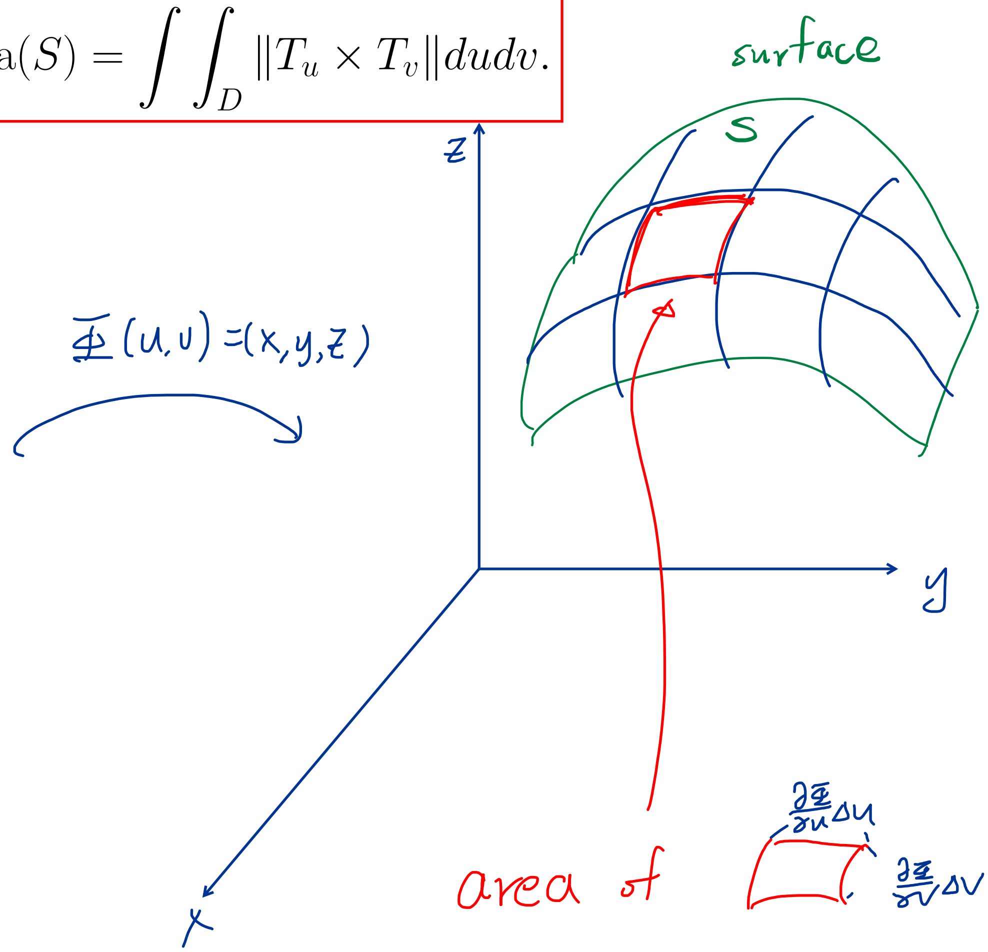
and

$$T_v = \frac{\partial \Phi}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Definition: (Area of a parametrized surface) We define the **surface area** (Area(S)) of a parametrized surface S by

$$\boxed{\text{Area}(S) = \int\int_D \|T_u \times T_v\| du dv.}$$





$\sim \left\| \frac{\partial \Phi}{\partial u} \Delta u \times \frac{\partial \Phi}{\partial v} \Delta v \right\|$

Example 4. *(Spring 2010) Let* $\Phi(u, v) = (u - v,\ u + v,\ uv)$ *and let* D *be the unit disk in the uv plane. Find the area of the surface* $S = \Phi(D)$.

$$T_u = \langle 1, 1, v \rangle$$

$$T_v = \langle -1, 1, u \rangle$$

$$T_u \times T_v = \langle u - v, -(v+u), 2 \rangle$$

$$\| T_u \times T_v \| = \sqrt{4 + 2u^2 + 2v^2}$$

$$\text{area}(S) = \iint_D \| T_u \times T_v \| \, du\, dv$$

$$= \iint_{u^2+v^2 \le 1} \sqrt{4 + 2u^2 + 2v^2} \, du\, dv$$

$u = r\cos\theta$

$v = r\sin\theta$

$$= \int_0^{2\pi}\int_0^1 \sqrt{4 + 2r^2}\, r\, dr\, d\theta$$

$w = 4 + 2r^2$

$dw = 4r\,dr$

$$= \int_0^{2\pi} \frac{1}{4}\frac{2}{3}\left(4 + 2r^2\right)^{\frac{3}{2}}\Big|_0^1 \, d\theta$$

$$= 2\pi\left(\sqrt{6} - \tfrac{4}{3}\right) \#$$

Remark:

In chapter 2 we have learned the tangent plane equations with respect to the level surface and the graph:

1. The tangent plane of the level surface $f(x, y, z) = c$ (c is a constant) at point (x_0, y_0, z_0) is

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

2. The tangent plane of the graph $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$ is

$$\left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0), -1\right) \cdot (x - x_0, y - y_0, z - f(x_0, y_0)) = 0.$$

$\underline{Ex}$: $z = \sqrt{x^2 + y^2}$

View it as a parametrization

$$\Phi(x,y) = (x, y, f(x,y)).$$

$$T_x = \langle 1, 0, f_x \rangle$$

$$T_y = \langle 0, 1, f_y \rangle$$

$$T_x \times T_y = \langle -f_x, -f_y, 1 \rangle.$$

7.5 Integrals of a real-valued function over a surface

Let D be an elementary region. Let $\Phi: D \to \mathbb{R}^3$ be written as

$$\Phi(u,v) = (x(u,v),\ y(u,v),\ z(u,v)),$$

which is a parametrization of the surface S.

Definition:
The integral of a real-valued function $f(x,y,z)$ over a surface S is defined as

$$\int\int_S f(x,y,z)dS = \int\int_D f(\Phi(u,v)) \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dudv. \tag{1}$$

Remark:

1. To find the area of S, we just take $f = 1$ in (1). Then

$$\text{Area}(S) = \int\int_D \left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| dudv.$$

2. In particular, if a surface S is the graph of a differentiable function $z = g(u,v)$, (u,v) in D. Then

$$\left\| \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \right\| = \sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}$$

and

$$\int\int_S f(x,y,z)dS = \int\int_D f(u,v,g(u,v))\sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}\ dudv. \tag{2}$$

$\Phi(u,v) = (u, v, g(u,v))$.

$T_u \times T_v = \langle -g_u, -g_v, 1 \rangle$.

Example 1. *Evaluate* $\int\int_S (1+4z)dS$, *where* S *is the surface* $x^2+y^2=z,\ 0 \leq z \leq 1$.

Parametrization: $\Phi(x,y) = (x, y, x^2+y^2)$.

$0 \leq x^2+y^2 \leq 1$.

$T_x = \langle 1, 0, 2x \rangle$ $\quad T_x \times T_y = \langle -2x, -2y, 1 \rangle$.

$T_y = \langle 0, 1, 2y \rangle$

$\| T_x \times T_y \| = \sqrt{4x^2+4y^2+1}$.

$$\iint (1+4z) \underline{\| T_x \times T_y \|}\, dx\, dy$$

$$= \iint \left(1+4(x^2+y^2)\right) \underline{\sqrt{4x^2+4y^2+1}}\, dx\, dy.$$

To be continued!