Spring 2018

Review: Stokes' theorem

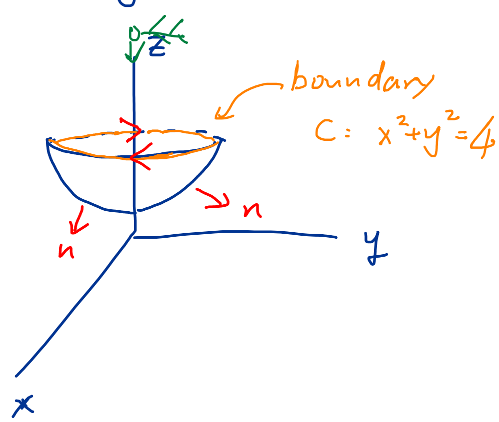
$$\int_C F \cdot ds = \iint_S (\nabla \times F) \cdot dS$$

where C is the oriented boundary of S.

EX: ① upper sphere


$$\int_C F \cdot ds \overset{\text{Stokes'}}{=} \iint_S (\nabla \times F) \cdot dS$$

② surface
$z = x^2 + y^2 + 1$, with $z \le 5$.

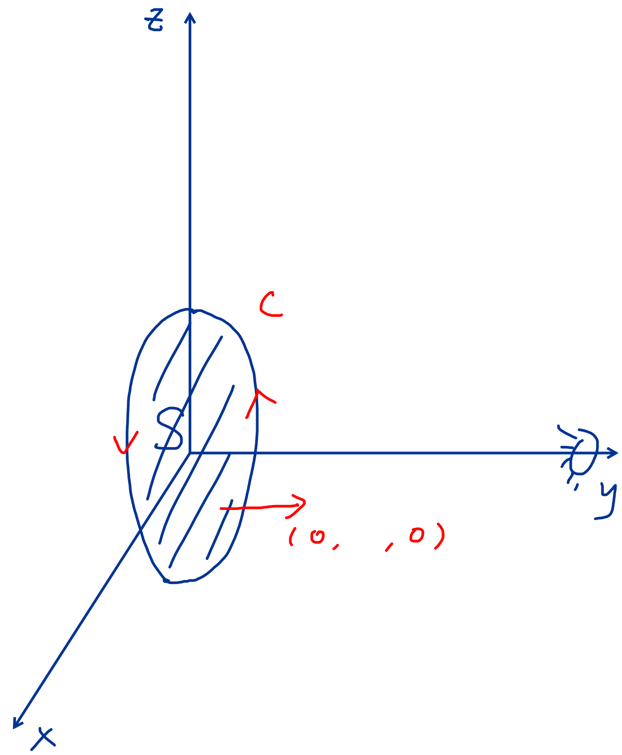


**Example 2.** *Let*

$$F(x,y,z) = (x^2 z + \sqrt{x^3 + x^2 + 2},\ xy,\ xy + \sqrt{z^3 + z^2 + 2}).$$

*Compute* $\int_c F \cdot d\mathbf{s}$, *where* $c$ *is the circle* $x^2 + z^2 = 1$, $y = 0$, *oriented in the counterclockwise direction when viewed from far out on the* $+y$*-axis.*

Note that: It is "much" easier to compute $\int_c F \cdot d\mathbf{s}$ by using Stokes' Theorem than by computing it directly.




By Stokes' theorem,

$$\int F \cdot ds = \iint_S (\nabla \times F) \cdot dS.$$

$\nabla \times F = \langle x,\ x^2 - y,\ y \rangle.$

choose $S$ to be a disk $x^2 + z^2 \le 1$, $y = 0$.

Parametrization

$\Phi(r,\theta) = (r\cos\theta,\ 0,\ r\sin\theta)$, $0 \le r \le 1$, $0 \le \theta < 2\pi$

$T_r = (\cos\theta,\ 0,\ \sin\theta)$

$T_\theta = (-r\sin\theta,\ 0,\ r\cos\theta)$

$T_r \times T_\theta = (0,\ -r,\ 0)$, points "$-y$" direction.

We adjust it by adding "$-$" sign into $\iint (\nabla \times F) \cdot dS$

$$-\iint (\nabla \times F) \cdot dS = -\int_0^{2\pi}\int_0^1 (r\cos\theta,\ r^2\cos^2\theta,\ 0) \cdot (0, -r, 0)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3 \cos^2\theta \, dr\, d\theta.$$

Recall $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$

$$= \int_0^{2\pi} \left( \frac{1}{4} r^4 \Big|_0^1 \right) \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} \cos^2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\, d\theta = \pi/4.$$

**Example 3.** *Let the surface $S$ be a portion of a sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$ and upward-pointing normal. Find*

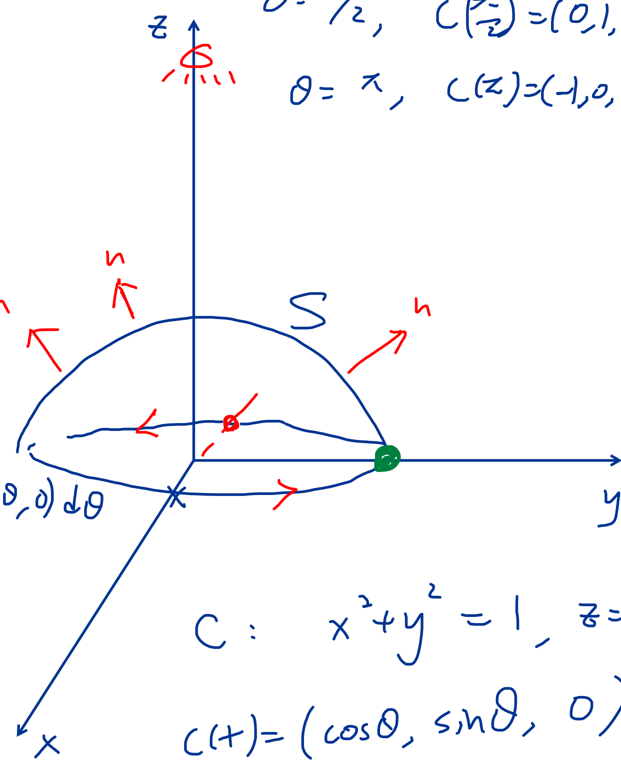
$$\int\int_S (\nabla \times F) \cdot d\mathbf{S},$$

*where $F = (y,\ -x,\ e^{xz} + xy^2z^5 + 1000)$.*

$\theta = 0$, $c(0) = (1, 0, 0)$

$\theta = \pi/2$, $c(\frac{\pi}{2}) = (0, 1, 0)$

$\theta = \pi$, $c(\pi) = (-1, 0, 0)$




By Stokes' theorem,

$$\iint_S (\nabla \times F) \cdot dS = \int_C F \cdot dS$$

$$= \int_0^{2\pi} \overbrace{(\sin\theta, -\cos\theta, 1 + 0 + 1000) \cdot (-\sin\theta, \cos\theta, 0)}^{-\sin^2\theta + (-\cos^2\theta) = -1} d\theta$$

$$= \int_0^{2\pi} -1 \; d\theta$$

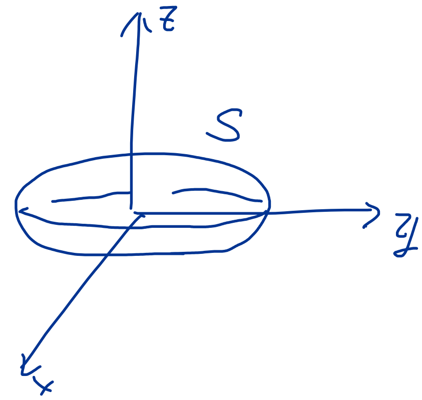
$$= \underline{-2\pi} \quad \#$$

$C:\ x^2 + y^2 = 1,\ z = 0.$

$c(t) = (\cos\theta, \sin\theta, 0)$

$0 \leq \theta < 2\pi.$

**Example 4.** *Find the surface integral* $\int\int_S (\nabla \times F) \cdot d\mathbf{S}$, *where* $S$ *is the ellipsoid* $x^2 + y^2 + 2z^2 = 10$ *and* $F = (\sin(xy),\ e^x,\ -yz)$. [1]




Stokes' theorem implies

$$\iint_S (\nabla \times F) \cdot dS = \int_C F \cdot ds.$$

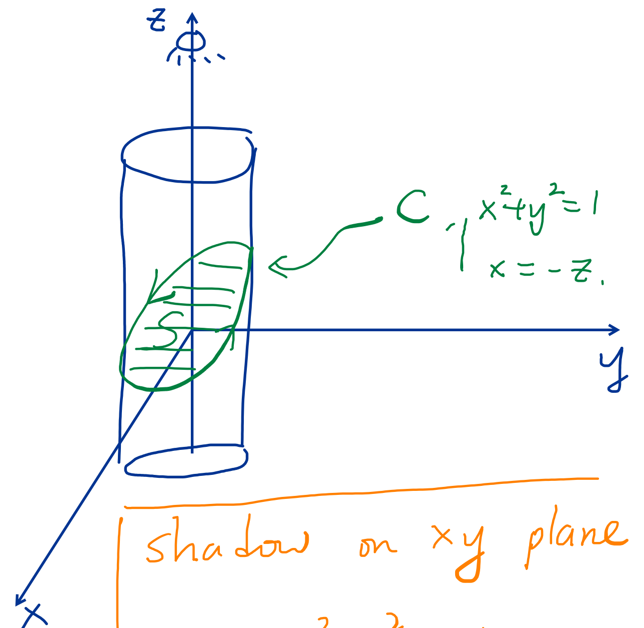
$$= 0.$$

[1] If $S$ has NO boundary, such as sphere, ellipsoid,..., etc., then $\int\int_S (\nabla \times F) \cdot d\mathbf{s} = 0$

**Example 5.** *Suppose a particle moves along the curve $C$ where $C$ is the curve formed by intersecting the cylinder $x^2+y^2=1$ with $x=-z$, oriented in the counterclockwise direction when viewed from high on the positive $z$ axis. Let*

$$F=\left(xy^2+\sqrt{x^4+1},\ 0,\ xy+\sqrt{z^3+z^2+2}\right)$$

*be the force acts on the particle. Find the work done on the particle by the force $F$ along the curve $C$.*




$$\text{work} = \int_C F \cdot d\vec{s}$$

By Stokes

$$= \iint_S (\nabla \times F) \cdot d\vec{S}$$

① $\nabla \times F = (x, -y, -2xy)$.

② Parametrize $S$ by

$\Phi(x,y) = (x, y, -x)$, $x^2+y^2 \le 1$

shadow on $xy$ plane is $x^2+y^2 \le 1$.

$T_x = (1, 0, -1)$

$T_y = (0, 1, 0)$

$T_x \times T_y = (1, 0, 1)$, pointing upward.

$$\iint_{x^2+y^2\le 1} (x, -y, -2xy) \cdot (1, 0, 1)\, dx\,dy$$

$$= \iint_{x^2+y^2\le 1} x - 2xy \ dx\,dy$$

$x = r\cos\theta$
$y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^1 (r\cos\theta - 2r^2\cos\theta\sin\theta)\, r\,dr\,d\theta$$

$$= 0. \ \#$$

**Example 6.** *Let* $F(x,y,z) = (\cos x \sin z + xy,\ x^3,\ e^{x^2+z^2} - e^{y^2+z^2} + \tan(xy))$.

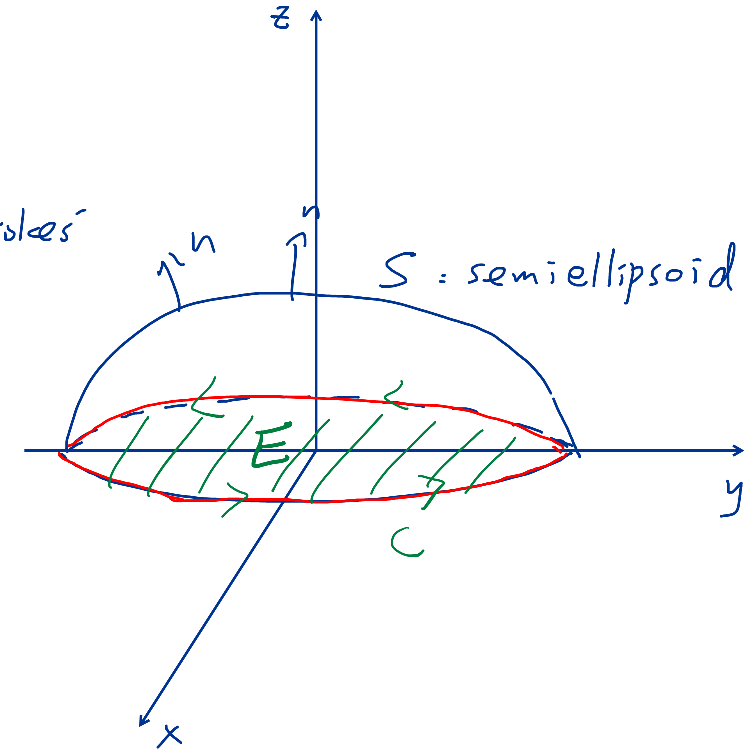
$$\int\int_S (\nabla \times F) \cdot d\mathbf{S},$$

*where* $S$ *is the semiellipsoid* $9x^2+4y^2+36z^2 = 36$, $z \geq 0$, *with upward-pointing normal.*

Stokes' theorem ,

$$\iint_S (\nabla \times F) \cdot dS = \int_C F \cdot ds \quad \Big\} \text{ Stokes'}$$

$$= \iint_E (\nabla \times F) \cdot dS$$




① $\nabla \times F = (\sim, \sim, 3x^2 - x)$.

② $E$ : $9x^2+4y^2 \leq 36$, $z=0$.

$$\frac{x^2}{4} + \frac{y^2}{9} \leq 1$$

$\sqrt{4}$ $\sqrt{9}$

$\Phi(r,\theta) = (2r\cos\theta,\ 3r\sin\theta,\ 0)$, $0 \leq r \leq 1$, $0 \leq \theta < 2\pi$.

$T_r \times T_\theta = \langle 0, 0, 6r \rangle$, pointing upward.

$$\iint_E (\nabla \times F) \cdot dS = \iint (\sim, \sim, 3x^2 - x)\Big|_{\Phi(r,\theta)} \cdot (0, 0, 6r)\, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 \left(3(2r\cos\theta)^2 - 2r\cos\theta\right) 6r \; dr\, d\theta$$

$$= \underline{18\pi} \ \#$$