§Find absolute maximum and minimum.

Suppose $f : K \to \mathbb{R}^1$ where K is a region in $\mathbb{R}^2$.

We say a point (x_0, y_0) in K is a **absolute maximum** (global) at f if

$$f(x,y) \leq f(x_0,y_0) \text{ for all } (x,y) \text{ in } K.$$

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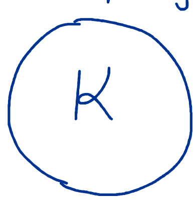
Closed and bounded sets:

- A set K is called **bounded** if it can be strictly contained in some bigger ball.
- A set K is called **closed** if it contains all its boundary points.

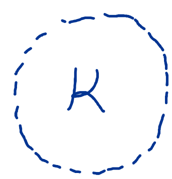
Example 3.

$x^2+y^2 \leq 1$.



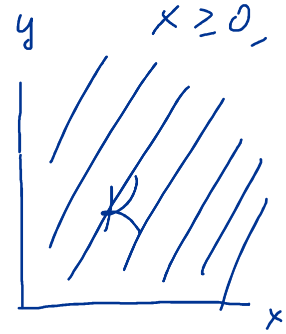
K is bounded. closed.

$x^2+y^2 < 1$



bounded. NOT closed.

$x \geq 0,\ y \geq 0$.



unbounded. closed.

Fact. *Suppose $f : K \to \mathbb{R}^1$ is continuous and K is a closed and bounded region in $\mathbb{R}^2$. Then f always has absolute (global) maximum and minimum at some points x_0 and x_1 of K.*

To find the absolute maximum and minimum of f in a closed and bounded region K.

Strategy:

$f(x,y)$. $\begin{cases} f_x = 0 \\ f_y = 0. \end{cases}$ find critical point (x,y)

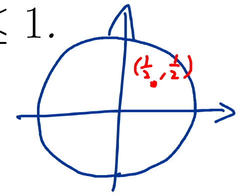
(a) Find all critical points in K and plug them into f.

(b) Find maximum and minimum on the boundary of K.

(c) Compare the values from (a) and (b).

The purpose to check the boundary is because global max / min can occur there.

Example 4. *Find the absolute maximum and absolute minimum values of* $f(x,y) = x^2 + y^2 - x - y + 1$ *in the disc* R *defined by* $x^2 + y^2 \leq 1$.



1. Critical points:

$$f_x = 2x - 1 = 0, \quad x = \frac{1}{2}$$

$$f_y = 2y - 1 = 0, \quad y = \frac{1}{2}.$$

$(\frac{1}{2}, \frac{1}{2})$.

$$f(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{1}{4} + \frac{1}{4} - \frac{1}{2} - \frac{1}{2} + 1 = \boxed{\frac{1}{2}}$$

absolute min. value

2. Consider f on the boundary $x^2+y^2=1$.

Parametrize $x^2+y^2=1$.

$C(t) = (\cos t, \sin t), \quad 0 \leq t < 2\pi$.

Plug into f by replacing $x = \cos t$, $y = \sin t$.

$$f(c(t)) = \underbrace{\cos^2 t + \sin^2 t}_{1} - \cos t - \sin t + 1$$

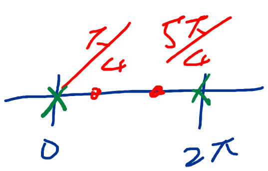
$$= 2 - \cos t - \sin t, \quad 0 \leq t < 2\pi$$

Let $g(t) = 2 - \cos t - \sin t$, on $[0, 2\pi]$.

$g'(t) = \sin t - \cos t, \quad [0, 2\pi]$.

critical point: $g'(t) = 0 \Rightarrow \sin t = \cos t$

$\Rightarrow t = \pi/4, \ 5\pi/4$.



Boundary point: $t = 0, 2\pi$.

$g(\pi/4) = 2 - \sqrt{2} \qquad g(0) = 1$

$g(5\pi/4) = \boxed{2 + \sqrt{2}} \qquad g(2\pi) = 1$

absolute maxi. value