**Example 3.** *Let*

$$f(x,y)=(x^2-3,3xy-y^3),\quad g(u,v)=(2uv,u^2-v^2,u+v).$$

*Find* $D(g\circ f)(1,2)$.

$Dg(f(1,2))\ Df(1,2)$.

$Df(x,y)=\begin{bmatrix}2x & 0\\ 3y & 3x-3y^2\end{bmatrix}$, $Df(1,2)=\begin{bmatrix}2 & 0\\ 6 & 3-12\end{bmatrix}=\begin{bmatrix}2 & 0\\ 6 & -9\end{bmatrix}$.

$Dg(u,v)=\begin{bmatrix}2v & 2u\\ 2u & -2v\\ 1 & 1\end{bmatrix}$, $f(1,2)=(1-3,\ 3\cdot 2-8)$

$=(-2,\ -2)$

$=(u,\ v)$

$Dg(-2,-2)=\begin{bmatrix}-4 & -4\\ -4 & 4\\ 1 & 1\end{bmatrix}$

$(-4,-4)\cdot(2,6)=-8-24$

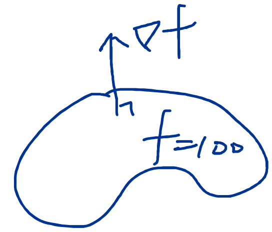
$D(g\circ f)(1,2)=Dg(f(1,2))\ Df(1,2)$

$=\begin{bmatrix}-4 & -4\\ -4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 6 & -9\end{bmatrix}=\begin{bmatrix}\boxed{-32} & 36\\ 16 & -36\\ 8 & -9\end{bmatrix}$. #

**Example 4.** *Consider the level surface which is given by equation $2xy + x^7 + 9y^3 + 2z = 100$. Find the equation for the tangent plane at the point $(x, y, z) = (2, 1, 0)$.*

Consider $f(x,y,z) = 2xy + x^7 + 9y^3 + 2z$.

Now we have $f = 100$.




Tangent plane :

$$\nabla f(2,1,0) \cdot (x-2,\ y-1,\ z-0) = 0.$$

$$\nabla f = (2y + 7x^6,\ 2x + 27y^2,\ 2).$$

$$\nabla f(2,1,0) = (2 + 7 \cdot 2^6,\ 4 + 27,\ 2)$$

$$= (450,\ 31,\ 2).$$

$$(450,\ 31,\ 2) \cdot (x-2,\ y-1,\ z) = 0.$$

$$450x + 31y + 2z \underbrace{-900 - 31}_{=\,-981} = 0$$

(Interchange the integration order.)

**Example 5.** *Evaluate the integral:*

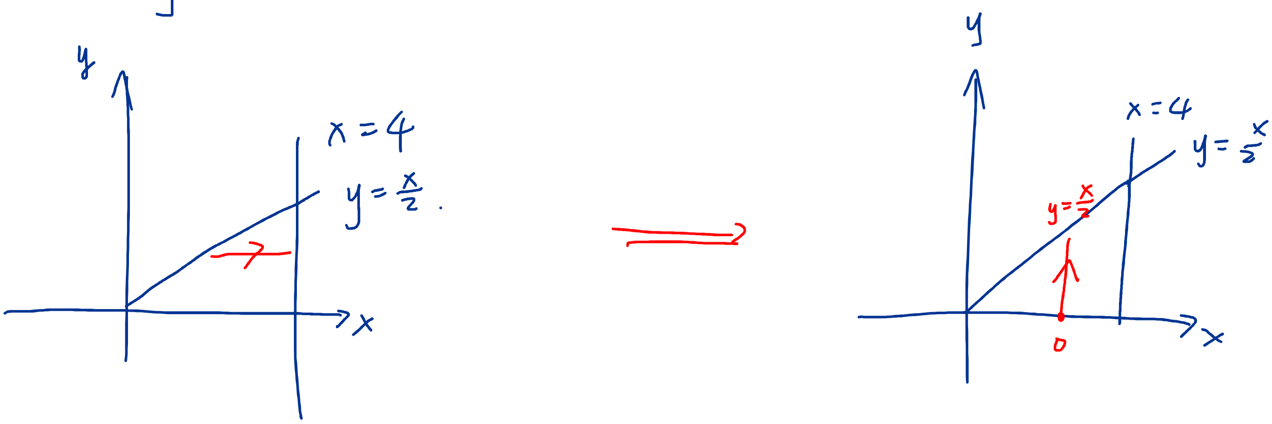
$$\int_0^2 \int_{2y}^4 \cos(x^2+1)\, dx dy.$$

$2y \le x \le 4 \quad \Rightarrow \quad 2y = x,\ x = 4.$

$0 \le y \le 2.$

$0 \le x \le 4$

$0 \le y \le x/2.$




$$\int_0^4 \int_0^{x/2} \cos(x^2+1)\, dy\, dx = \int_0^4 \cos(x^2+1)\, \frac{x}{2}\, dx$$

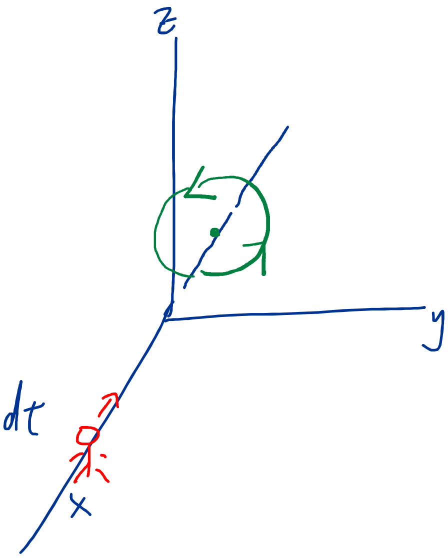
$u = x^2 + 1$

$$= \frac{1}{4} \sin(x^2+1) \Big|_0^4$$

$$= \frac{1}{4}\Big(\sin(17) - \sin(1)\Big).$$

**Example 6.** *Let $C$ is the intersection of cylinder $y^2+z^2=1$ with the plane $x=-1$, oriented in the counterclockwise direction when viewed from far out the positive x axis. Let*

$$F = (\sqrt{x^3+y^3+5}, z, x^2)$$




1. *Evaluate $\int_C F \cdot d\mathbf{s}$.*

$C(t) = (-1, \cos t, \sin t), \quad 0 \le t \le 2\pi.$

$$\int F \cdot d\vec{s} = \int F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^{2\pi} \left(\sqrt{-1+\cos^3 t+5},\ \sin t,\ 1\right) \cdot (0, -\sin t, \cos t)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos t \quad dt$$

$$= \int_0^{2\pi} -\frac{1-\cos(2t)}{2} + \cos t \quad dt$$

$$= -\pi. \quad \#$$

2. *Suppose that the curve $C$ represent a wire with density $\mu(x,y,z) = x^2+y^2$ grams per unit length of wire. Compute the total mass of the wire.*

$$\text{Total mass} = \int \mu(c(t))\, \|c'(t)\|\, dt$$

$$= \int (1+\cos^2 t)\, \underbrace{\|(0, -\sin t, \cos t)\|}_{1}\, dt$$

$$= \int_0^{2\pi} 1+\cos^2 t \quad dt$$

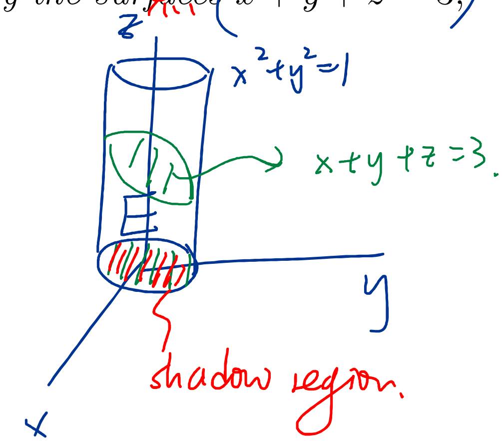
$$= \int_0^{2\pi} 1+\frac{1+\cos(2t)}{2} \quad dt.$$

$$= \underline{3\pi} \quad \#$$

(Compute the volume: Triple integrals)

**Example 7.** *Find the volume of the solid $E$ which is bounded by the surfaces $x + y + z = 3$, $z = 0$ and $x^2 + y^2 = 1$.*

$0 \le z \le 3 - x - y.$




$$\iint_{x^2+y^2\le 1} \int_0^{3-x-y} 1 \, dz \, dx\,dy$$

$$= \iint_{x^2+y^2\le 1} (3 - x - y) \, dx\,dy$$

$x^2+y^2 \le 1.$

polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases} \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

$$= \int_0^{2\pi} \int_0^1 (3 - r\cos\theta - r\sin\theta) \, r \, dr\,d\theta.$$

$$= 3\pi.$$