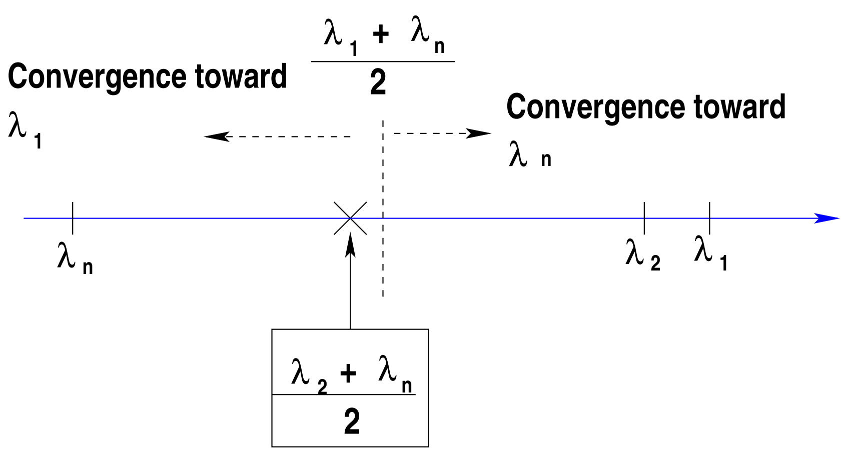
✍1 Convergence factor $\phi(\sigma)$ as a function of σ.

Solution: The eigenvalues of the shifted matrix are $\lambda_i - \sigma$. When $\sigma > (\lambda_1 + \lambda_n)/2$ then the algorithm will converge toward λ_n because $|\lambda_n - \sigma| > |\lambda_1 - \sigma|$. We will ignore this case.

Assume now that $\sigma < (\lambda_1 + \lambda_n)/2$. If $\sigma < (\lambda_2 + \lambda_n)/2$ then largest eigenvalue of $A - \sigma$ is $\lambda_1 - \sigma$ and second largest is $\lambda_2 - \sigma$. If $\sigma \geq (\lambda_2 + \lambda_n)/2$ then largest eigenvalue of $A - \sigma$ is $\lambda_n - \sigma$ and second largest is $\lambda_2 - \sigma$. Therefore, setting $\mu = (\lambda_2 + \lambda_n)/2$, we get

$$\phi(\sigma) = \begin{cases} \frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|} = \frac{\lambda_2 - \sigma}{\lambda_1 - \sigma} & \text{if } \sigma < \mu \\ \frac{|\lambda_n - \sigma|}{|\lambda_1 - \sigma|} = \frac{\sigma - \lambda_n}{\lambda_1 - \sigma} & \text{if } \sigma > \mu \end{cases}$$



Note that for $\sigma < \mu$ we have $\phi(\sigma) = 1 - (\lambda_1 - \lambda_2)/(\lambda_1 - \sigma)$ which is a decreasing function while when $\sigma > \mu$ we have $\phi(\sigma) = -1 + (\lambda_1 - \lambda_n)/(\lambda_1 - \sigma)$ which is an increasing function. The min. is reached when these 2 values are equal which leads to the solution $\sigma_{opt} = (\lambda_n + \lambda_2)/2$ $\square$

Jacobi method:

✍2 Let $\|A_O\|_I = \max_{i\neq j} |a_{ij}|$. Show that

$$\|A_O\|_F \le \sqrt{n(n-1)}\|A_O\|_I$$

Solution: This is straightforward:

$$\|A_O\|_F^2 = \sum_{i\neq j} |a_{ij}|^2 \le n(n-1) \max_{i\neq j} |a_{ij}|^2 = n(n-1)\|A_O\|_I^2.$$

$\square$

✍3 Use this to show convergence in the case when largest entry is zeroed at each step.

Solution: If we call B_k the matrix A_O after each rotation then we have according to result in the previous page and using the previous exercise:

$$\begin{aligned}\|B_{k+1}\|_F^2 &= \|B_k\|_F^2 - 2a_{pq}^2 \\ &= \|B_k\|_F^2 - 2\|B_k\|_I^2 \\ &\leq \|B_k\|_F^2 - \frac{2}{n(n+1)}\|B_k\|_F^2 \\ &= \left[1 - \frac{2}{n(n+1)}\right] \|B_k\|_F^2\end{aligned}$$

which shows that the norm will be decreasing by factor less than a constant that is less than one - therefore it converges to zero. $\square$

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Notes on the QR method

It is important to understand the reasoning for the methods discussed here. If you apply the basic QR algorithm in its basic form you will get a cost that is like

$$n_{it}\, n^3$$

The problem is that n_{it} the number of iterations is unknown and can be quite large.

The next observation from the theory is that the last row converges faster. In addition its convergence is quadratic if we use shifts or origin. This makes the algorithm much more interesting. Quadratic convergence means that for all practical purposes a few steps per row. Each of these steps costs $O(n^3)$ and so we now have a cost of $O(n^4)$. The last improvement - comes from the use of the Hessenberg form. Each of the $O(n^3)$ operations becomes $O(n^2)$ and this results in a total cost of $O(n^3)$.

These developments took decades to unravel completely. If you call the function eig from Matlab, it will compute eigenvalues with the QR algorithm. It is important to know what is done behind the scenes.