✍1 Exact solution of system

$$\begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$$

Solution: You will find $x = [1, 3, -2]^T$. □

✍2 Justify the column version of Back-subsitution algorithm.

Solution: The system $Ax = b$ can be written in column form as follows:

$$x_1 a_{:,1} + x_2 a_{:,2} + \cdots + x_n a_{:,n} = b$$

In first step we compute $x_n = b_n / a_{n,n}$. Now move last term in left-hand side of above system to the right:

$$x_1 a_{:,1} + x_2 a_{:,2} + \cdots + x_{n-1} a_{:,n-1} = b - x_n a_{:,n} \equiv b^{(1)}$$

This is a new system of n equations that has $(n-1)$ unknowns and the right-hand-side $b^{(1)}$. The last equation of this system is of the form $0 = 0$ and can therefore be ignored. Thus, we end up wih a system of size $(n-1)\times(n-1)$ that is still upper triangular and we can repeat the above argument recursively. $\square$

✍3 Exact operation count for GE.

Solution:

$$
\begin{aligned}
T &= \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} (2(n-k)+3) \\
&= \sum_{k=1}^{n-1} (2(n-k)+3)(n-k) \\
&= 2\sum_{k=1}^{n-1}(n-k)^2 + 3\sum_{k=1}^{n-1}(n-k) \\
&= 2\sum_{j=1}^{n-1} j^2 + 3\sum_{j=1}^{n-1} j
\end{aligned}
$$

In the last step we made a change of variables $j = n-k$. Now we know that $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ and $\sum_{k=1}^{n} k = n(n+1)/2$ and so

$$
\begin{aligned}
T &= 2\frac{(n-1)(n)(2n-1)}{6} + 3\times\frac{n(n-1)}{2} \\
&= \ldots. \\
&= n(n-1)\left(\frac{2n}{3}+\frac{7}{6}\right) \qquad (1)
\end{aligned}
$$

Note in passing the remarkable fact that the above final expression is always an integer (it has to be) no matter what (integer) value n takes.

□

✍4 Practical use: Show how to use the LU factorization to solve linear systems with the same matrix A and different b's.

Solution: If we have the LU factorization $A = LU$ available then we can solve the linear system $Ax = b$ by writing it as

$$L \underbrace{(Ux)}_{y} = b$$

So we solve for y: $Ly = b$ then once y is computed we solve for x: $Ux = y$. This involves two triangular solves at the cost of n^2 each instead of the $O(n^3)$ cost of redoing everything with Gaussian elimination.□

✍5 LU factorization of the matrix $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix}$?

Solution: You will find

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & -7/3 \end{pmatrix} \qquad \square$$

✍6 Determinant of A?

Solution: It is the determinant of U which is -12.

✍7 True or false: "Computing the LU factorization of matrix A involves more arithmetic operations than solving a linear system $Ax = b$ by Gaussian elimination".

Solution: The number of arithmetic operations is identical. (The LU factorization involves additional memory moves to store the factors - but these are no floating point operations) ☐

✍8 Operation count for Gauss-Jordan. Order of the cost? How does it compare with Gaussian Elimination?

Solution: From the notes:

$$
\begin{aligned}
T &= \sum_{k=1}^{n-1}\sum_{i=1}^{n-1}[2(n-k)+3)] = \sum_{k=1}^{n-1}(n-1)[2(n-k)+3] \\
&= (n-1)\sum_{j=1}^{n-1}[2j+3] \\
&= (n-1)\left[n(n-1)+3(n-1)\right] \\
&= (n-1)^2(n+3) = (n-1)^3 + 4(n-1)^2
\end{aligned}
$$

The bottom line is that the cost is $\approx n^3$ which is 50% more expensive than GE. This additional cost is not worth it in spite of the simplicity of the algorithm. For this Gauss-Jordan is seldom used in practice. □

✍9 What is the matrix PA when

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 0 & -1 & 2 \\ -3 & 4 & -5 & 6 \end{pmatrix} ?$$

Solution: Instead of multiplying you just permute the row: row 1 in new matrix is row 3 of old matrix, row 2 is row 1 of old matrix, etc.

$$PA = \begin{pmatrix} 9 & 0 & -1 & 2 \\ 1 & 2 & 3 & 4 \\ -3 & 4 & -5 & 6 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

✍10 In the previous example where

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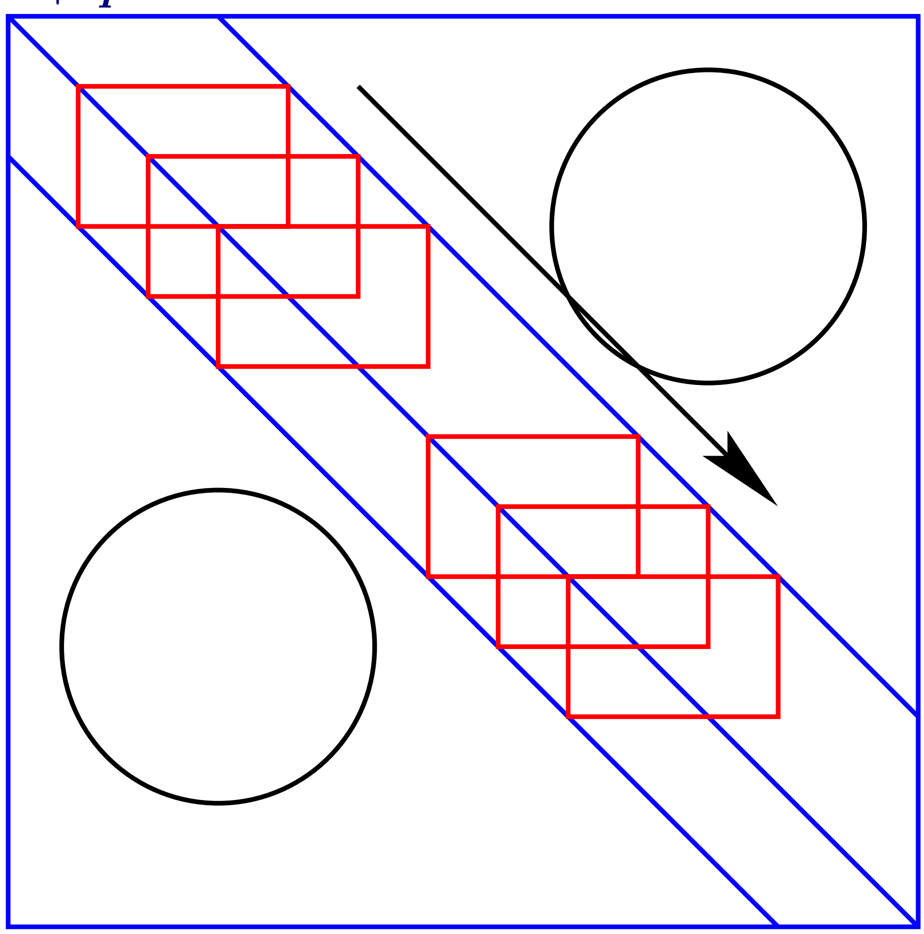
>> A = [ 1 2 3 4; 5 6 7 8; 9  0 -1 2 ; -3 4 -5 6]
```

Matlab gives $det(A) = -896$. What is $det(PA)$?

Solution: It changes sign so $\det(PA) = 896$. This is because the permutation $\pi = [3, 1, 4, 2]$ is made of 3 interchanges. □

✍11 Given a banded matrix with upper bandwidth q and lower bandwidth p, what is the operation count (leading term only) for solving the linear system $Ax = b$ with Gaussian elimination without pivoting? What happens when partial pivoting is used? Give the new operation count for the worst case scenario.

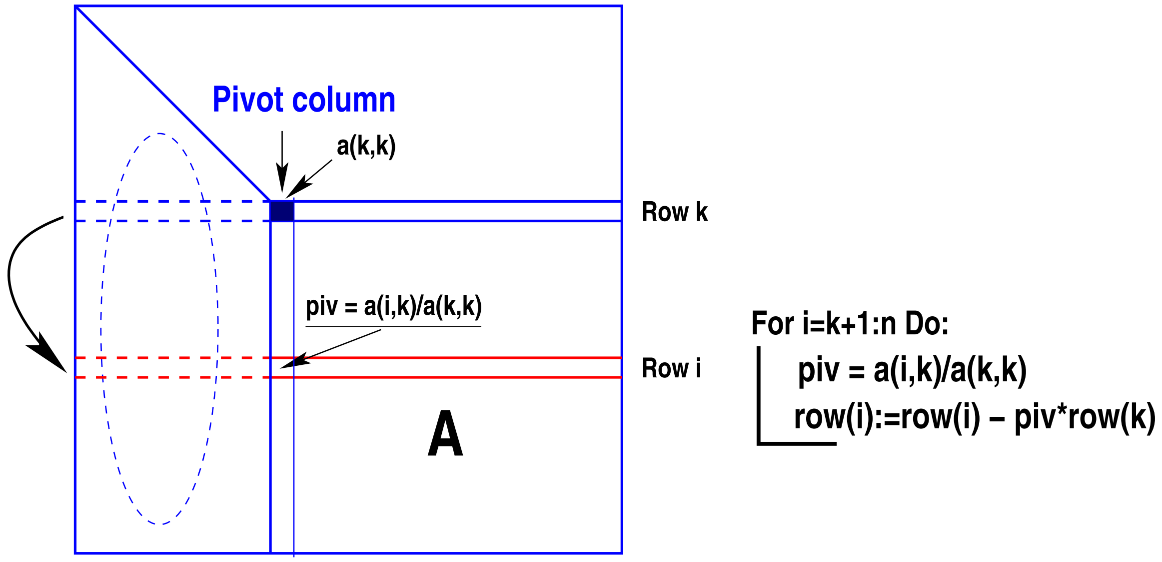
Solution: [Note: it is assumed that $p \ll n$ and $q \ll n$ but p and q are not related]. The important observation here is that Gaussian elimination without pivoting for this band matrix will operate on a rectangle: at step k only rows $k+1$ to $k+p$ are affected and columns $k+1$ to $k+q$ are affected.



In this rectangle each entry will be modified at the cost of 2 operations (*, +). Total: $2pq$ for each step. So Gaussian elimination without pivoting for this band matrix costs *approximately* $2npq$ flops. Using band backward substitution to obtain the solution x costs $\approx 2nq$ flops. The total operation count (leading term only): $\approx 2npq + 2nq = 2nq(p+1)$. Note that when p is small the cost of susbstitution cannot be ignored.

For the Gaussian elimination with pivoting, the upper bandwidth of the resulting matrix will be $p + q$. In this case, the total operation count (leading term only) becomes: $\approx 2np(p+q)(p+1)$. $\square$

Additional notes on the LU factorization The lecture notes mention an 'algorithmic' approach to understanding the LU factorization. Here again is the illustration of the k-th step of Gaussian Elimination (GE):



We will focus on how the i-th row is transformed throughout the algorithm. In words: the ith row undergoes $i-1$ transformations (each indexed by k in the algorithm). After these $i-1$ GE steps the row remains unchanged. Each of these $i-1$ transformations - which corresponds to the steps $k=1,2,\cdots,i-1$, is as follows

$$a_{i,:} = a_{i,:} - piv * a_{k,:}$$

We will need to make the following changes to the notation for better clarity. Once a row say $a_{j,:}$ no longer changes [i.e., when it undergoes no further transformations] we will call it $u_{j,:}$ - reflecting the fact that this will end up in the final U matrix of the LU factorization. In addition we will change 'piv' in the above equation into l_{ik} which we recall is equal to a_{ik}/a_{kk}. Finally, we must also add a superscript to row i to index the transformation number k. With this, the above equation becomes

$$a_{i,:}^{(k)} = a_{i,:}^{(k-1)} - l_{ik} * u_{k,:}$$

Notice how $a_{k,:}$ has been changed to $u_{k,:}$. Indeed the pivot row used for any elimination no longer changes. We will write the above relation for $k=1,2,\cdots,i-1$. After these $i-1$ transformations $a_{i,:}^{(k)}$ is no longer changed and becomes the row u_k, the kth row of U.

$$a_{i,:}^{(1)} = a_{i,:} - l_{i1} * u_{1,:}$$
$$a_{i,:}^{(2)} = a_{i,:}^{(1)} - l_{i2} * u_{2,:}$$
$$a_{i,:}^{(3)} = a_{i,:}^{(2)} - l_{i3} * u_{2,:}$$
$$\cdots = \cdots - () * (\cdots)$$
$$a_{i,:}^{(i-2)} = a_{i,:}^{(i-3)} - l_{i,i-2} * u_{2,:}$$
$$a_{i,:}^{(i-1)} = a_{i,:}^{(i-2)} - l_{i,i-1} * u_{2,:}$$

Notice that $a_{i:}^{(0)}$ is just $a_{i:}$. If you add all the equations on the left - things cancel out - and you will wind up with:

$$a_{i:}^{(i-1)} = a_{i:} - \sum_{k=1}^{i-1} l_{ik} u_{k:}$$

The row $a_{i:}^{(i-1)}$ is no longer modified.

Therefore, it should be change to $u_{i:}$. and so we get:

$$u_{i:} = a_{i:} - \sum_{k=1}^{i-1} l_{ik} u_{k:} \quad \text{or} \quad a_{i:} = u_{i:} + \sum_{k=1}^{i-1} l_{ik} u_{k:}.$$

Next define the matrix L whose entries l_{ij}'s are the same as above for $i > j$ (lower part), $l_{ii} = 1$ (diagonal), and $l_{ij} = 0$ for $j > i$ (upper part). The above equation can now be rewritten as

$$a_{i:} = u_{i:} + \sum_{k=1}^{i-1} l_{ik} u_{k:} = \sum_{k=1}^{n} l_{ik} u_{k:}$$

This translates exactly the equation $A = LU$ written in row-form. □